An Abstract Completion Procedure for Cut Elimination in Deduction Modulo

LICS 2006

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Sequent Calculus Modulo

Deduction modulo $=$ deduction $+$ calculus (Poincaré's Principle)

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Calculus: conversion rules

$$\frac{\Gamma, B \vdash \Delta}{\Gamma, A \vdash \Delta}\ \text{Conv-l} \qquad \text{if } A \equiv B \qquad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A, \Delta}\ \text{Conv-r}$$

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Calculus: conversion rules

$$\frac{\Gamma, B \vdash \Delta}{\Gamma, A \vdash \Delta} \uparrow\text{-l} \quad \text{if } A \rightarrow B \quad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A, \Delta} \uparrow\text{-r}$$

Here: rewriting atomic propositions to propositions

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Can we prove every valid sequent without the cut rule ?

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Congruence [Crabbé, 1974]: $A \to B \wedge \neg A$

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There is no cut-free proof of $B \vdash$ (no inference rules to be applied)

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Using the Abstract Canonical Systems framework [Dershowitz and Kirchner, 2006]: proof ordering $\Rightarrow$ critical proofs $\Rightarrow$ abstract completion procedure $\Rightarrow$ completeness result

Theorem: Deduction modulo is an instance of Abstract Canonical System

http://www.loria.fr/~burel/download/gencomp.pdf

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- extends this to the whole first-order logic
- implementation (ML or TOM or Coq ?)

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