Exercise - answer

$$
\begin{aligned}
\text{maximize} \quad & 2x_1 + 4x_2 + x_3 \\
\text{subject to:} \quad & 2x_1 + x_2 + x_3 \leq 10 && (1)\\
& x_1 + x_2 - x_3 \leq 4 && (2)\\
& 0 \leq x_1 \leq 4,\ 0 \leq x_2 \leq 6,\ 1 \leq x_3 \leq 6 && (3)
\end{aligned}
$$

The restricted master problem is given by the constraints (1) and (2) and the constraints at (3) will be at the Auxiliary Problem. Use $x_1 = x_2 = 0,\ x_3 = 1$, which means the column $(0, 0, 1)$ to start the solution of this exercise.

Restricted Master Problem

$$
\begin{aligned}
\text{maximize} \quad & z = \sum_{j=1}^{p_R} (c^\top v_j)\lambda_j && (4)\\
\text{subject to:} \quad & \sum_{j=1}^{p_R} (A_1 v_j)\lambda_j \leq 10 && (5)\\
& \sum_{j=1}^{p_R} (A_2 v_j)\lambda_j \leq 4 && (6)\\
& \sum_{j=1}^{p_R} \lambda_j = 1 && (7)
\end{aligned}
$$

Consider π_1, π_2 e π the dual variables related to the constraints (5), (6) and (7) of the Restricted Master Problem respectively. p_R are the columns of the restricted master problem.

Auxiliary Problem

$$
\begin{aligned}
\text{maximize} \quad & (2 - 2\pi_1 - \pi_2)x_1 + (4 - \pi_1 - \pi_2)x_2 + (1 - \pi_1 + \pi_2)x_3 - \nu \\
\text{subject to:} \quad & 0 \leq x_1 \leq 4,\ 0 \leq x_2 \leq 6,\ 1 \leq x_3 \leq 6
\end{aligned}
$$

First iteration

Let $x_1 = x_2 = 0,\ x_3 = 1$, which means the column $(0, 0, 1)$, as an initial feasible column. We are going to find the coefficients of the objective function and the constraints of RMP of the first iteration.

$$
c^\top v_1 = (2, 4, 1)\begin{pmatrix}0\\0\\1\end{pmatrix} = 1,\ A_1^\top v_2 = (2, 1, 1)\begin{pmatrix}0\\0\\1\end{pmatrix} = 1,\ A_2^\top v_2 = (1, 1, -1)\begin{pmatrix}0\\0\\1\end{pmatrix} = -1
$$

Thus, the the first Restricted Master Problem (RMP1) is:

$$
\begin{aligned}
(RMP1) \text{ maximize } & \; z = 1\lambda_1 \\
\text{subject to: } & \; 1\lambda_1 \leq 10 \\
& -1\lambda_1 \leq 4 \\
& \lambda_1 = 1
\end{aligned}
$$

$\overline{z} = 1$, $\lambda_1 = 1$ e $\pi_1 = \pi_2 = 0$ and $\nu = 1$.

Auxiliary problem 1 (AP1)

$$
\begin{aligned}
(AP1) \text{ maximize } & \; 2x_1 + 4x_2 + 1x_3 - 1 \\
\text{subject to: } & \; 0 \leq x_1 \leq 4,\; 0 \leq x_2 \leq 6,\; 1 \leq x_3 \leq 6
\end{aligned}
$$

OF $= 38$, $x_1 = 4$, $x_2 = x_3 = 6$. UB $= 1 + 38 = 39$.
Solving auxiliary problem for column 1, we have column 2,

$$v_2^\top = (4, 6, 6)$$

and the coefficients for λ_2 at the objective function of the master problem of the second iteration are:

$$c^\top v_2 = (2, 4, 1)\begin{pmatrix} 4 \\ 6 \\ 6 \end{pmatrix} = 38,\; a_1^\top v_2 = (2, 1, 1)\begin{pmatrix} 4 \\ 6 \\ 6 \end{pmatrix} = 20,\; a_2^\top v_2 = (1, 1, -1)\begin{pmatrix} 4 \\ 6 \\ 6 \end{pmatrix} = 4$$

Second iteration

$$
\begin{aligned}
(RMP2) \text{ maximize } & \; z = 1\lambda_1 + 38\lambda_2 \\
\text{subject to: } & \; 1\lambda_1 + 20\lambda_2 \leq 10 \\
& -1\lambda_1 + 4\lambda_2 \leq 4 \\
& \lambda_1 + \lambda_2 = 1
\end{aligned}
$$

$\overline{z} = 18.526316, \lambda_1 = 0.526316,\; \lambda_2 = 0.473684, \pi_1 = 1.947368,\; \pi_2 = 0,\; \nu = -0.947368$

Auxiliary problem

$$
\begin{aligned}
\text{maximize } & \; (2 - 2\pi_1 - \pi_2)x_1 + (4 - \pi_1 - \pi_2)x_2 + (1 - \pi_1 + \pi_2)x_3 - \nu \\
\text{subject to: } & \; 0 \leq x_1 \leq 4,\; 0 \leq x_2 \leq 6,\; 1 \leq x_3 \leq 6
\end{aligned}
$$

Replacing the values of $\pi_1 = 1.947368,\; \pi_2 = 0,\; \nu = -0.947368$ we have:

Auxiliary problem 2

$$
\begin{aligned}
(AP2) \text{ maximize } & \; -1.894736x_1 + 2.052632x_2 - 0.947368x_3 + 0.947368 \\
\text{subject to: } & \; 0 \leq x_1 \leq 4,\; 0 \leq x_2 \leq 6,\; 1 \leq x_3 \leq 6
\end{aligned}
$$

OF $= 12.315791999999998$, $x_1 = 0$, $x_2 = 6$, $x_3 = 1$.
UB $= 18.526316 + 12.315791999999998 = 30.842108$.

Column 3 is: $v_3^\top = (0, 6, 1)$ and the coefficients of λ_3 at the objective function and at the constraints are, respectively:

$$c^\top v_3 = (2,4,1)\begin{pmatrix}0\\6\\1\end{pmatrix} = 25,\ a_1^\top v_3 = (2,1,1)\begin{pmatrix}0\\6\\1\end{pmatrix} = 7,\ a_2^\top v_3 = (1,1,-1)\begin{pmatrix}0\\6\\1\end{pmatrix} = 5.$$

Third iteration

$$\begin{aligned} (RMP3)\ \text{maximize} \quad & z = 1\lambda_1 + 38\lambda_2 + 25\lambda_3 \\ \text{subject to:} \quad & 1\lambda_1 + 20\lambda_2 + 7\lambda_3 \le 10 \\ & -1\lambda_1 + 4\lambda_2 + 5\lambda_3 \le 4 \\ & \lambda_1 + \lambda_2 + \lambda_3 = 1 \end{aligned}$$

$\overline{z} = 25.857143$, $\lambda_1 = 0.119048$, $\lambda_2 = 0.285714$, $\lambda_3 = 0.595238$
$\pi_1 = 1.214286$, $\pi_2 = 2.785714$ and $\nu = 2.571429$

Auxiliary problem

$$\begin{aligned} \text{maximize} \quad & (2 - 2\pi_1 - \pi_2)x_1 + (4 - \pi_1 - \pi_2)x_2 + (1 - \pi_1 + \pi_2)x_3 - \nu \\ \text{subject to:} \quad & 0 \le x_1 \le 4,\ 0 \le x_2 \le 6,\ 1 \le x_3 \le 6 \end{aligned}$$

Replacing the values of $\pi_1 = 1.214286$, $\pi_2 = 2.785714$ and $\nu = 2.571429$ we have:

Auxiliary problem 3

$$\begin{aligned} (AP3)\ \text{maximize} \quad & -3.214286x_1 + 0.571428x_3 - 2.571429 \\ \text{subject to:}\ & 0 \le x_1 \le 4,\ 0 \le x_2 \le 6,\ 1 \le x_3 \le 6 \end{aligned}$$

OF $=0.857139$, $x_1 = x_2 = 0$, $x_3 = 6$.
UB: $25.857143 + 0.857139 = 26.714282$.
Column 4 is $v_4^\top = (0,0,6)$ and the coefficients for λ_4 at the objective function and at the constraints of the master problem are respectively:

$$c^\top v_4 = (2,4,1)\begin{pmatrix}0\\0\\6\end{pmatrix} = 6,\ a_1^\top v_4 = (2,1,1)\begin{pmatrix}0\\0\\6\end{pmatrix} = 6,\ a_2^\top v_4 = (1,1,-1)\begin{pmatrix}0\\0\\6\end{pmatrix} = -6.$$

Fourth iteration

$$\begin{aligned} (RMP4)\ \text{maximize} \quad & z = 1\lambda_1 + 38\lambda_2 + 25\lambda_3 + 6\lambda_4 \\ \text{subject to:} \quad & 1\lambda_1 + 20\lambda_2 + 7\lambda_3 + 6\lambda_4 \le 10 \\ & -1\lambda_1 + 4\lambda_2 + 5\lambda_3 - 6\lambda_4 \le 4 \\ & \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1 \end{aligned}$$

$\overline{z} = 26.75$, $\lambda_1 = 0.0$, $\lambda_2 = 0.236111$, $\lambda_3 = 0.694444$, $\lambda_4 = 0.069444$
$\pi_1 = 1.125$, $\pi_2 = 1.625$ and $\nu = 9.0$

Auxiliary problem

$$\begin{aligned} \text{maximize} \quad & (2 - 2\pi_1 - \pi_2)x_1 + (4 - \pi_1 - \pi_2)x_2 + (1 - \pi_1 + \pi_2)x_3 - \nu \\ \text{subject to:} \quad & 0 \le x_1 \le 4,\ 0 \le x_2 \le 6,\ 1 \le x_3 \le 6 \end{aligned}$$

Replacing the values of $\pi_1 = 1.125$, $\pi_2 = 1.625$ and $\nu = 9.0$ we have:

Auxiliary problem 4

$$\begin{aligned}(AP4)\ \text{maximize}\quad & -1.875x_1 + 1.25x_2 + 1.5x_3 - 9\\ \text{subject to: } & 0 \le x_1 \le 4,\ 0 \le x_2 \le 6,\ 1 \le x_3 \le 6\end{aligned}$$

OF=7.5, $x_1 = 0$, $x_2 = 6$, $x_3 = 6$
UB: 34.25.
Column 5 is $v_5^\top = (0, 6, 6)$ and the coefficients for λ_5 at the objective function and at the constraints of the master problem are respectively:

$$cv_5 = (2,4,1)^\top \begin{pmatrix} 0\\ 6\\ 6\end{pmatrix} = 30,\ A_1v_5 = (2,1,1)^\top \begin{pmatrix} 0\\ 6\\ 6\end{pmatrix} = 12,\ A_2v_5 = (1,1,-1)^\top \begin{pmatrix} 0\\ 6\\ 6\end{pmatrix} = 0.$$

Fifth iteration

$$\begin{aligned}(RMP5)\text{maximize } z &= 1\lambda_1 + 38\lambda_2 + 25\lambda_3 + 6\lambda_4 + 30\lambda_5\\ \text{subject to: } & 1\lambda_1 + 20\lambda_2 + 7\lambda_3 + 6\lambda_4 + 12\lambda_5 \le 10\\ & -1\lambda_1 + 4\lambda_2 + 5\lambda_3 - 6\lambda_4 + 0\lambda_5 \le 4\\ & \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 = 1\end{aligned}$$

$\overline{z} = 28.0$, $\lambda_1 = 0.0$, $\lambda_2 = 0.16666$, $\lambda_3 = 0.66666$, $\lambda_4 = 0.0$, $\lambda_5 = 0.16666$
$\pi_1 = 1.0$, $\pi_2 = 0.0$ and $\nu = 18.0$.

Auxiliary problem

$$\begin{aligned}\text{maximize}\quad & (2 - 2\pi_1 - \pi_2)x_1 + (4 - \pi_1 - \pi_2)x_2 + (1 - \pi_1 + \pi_2)x_3 - \nu\\ \text{subject to:}\quad & 0 \le x_1 \le 4,\ 0 \le x_2 \le 6,\ 1 \le x_3 \le 6\end{aligned}$$

Replacing the values of $\pi_1 = 1.0$, $\pi_2 = 0.0$ and $\nu = 18.0$ we have:

Auxiliary problem 5

$$\begin{aligned}(AP5)\ \text{maximize}\quad & 3x_2 - 18\\ \text{subject to: } & 0 \le x_1 \le 4,\ 0 \le x_2 \le 6,\ 1 \le x_3 \le 6\end{aligned}$$

OF=0, $x_1 = 0$, $x_2 = 6$, $x_3 = 1$.
UB: 28.0 .
As UB $= \overline{z} = 28.0$ we are at the optimal solution, that is, $\lambda_1 = 0.0$, $\lambda_2 = 0.166666666$, $\lambda_3 = 0.66666666$, $\lambda_4 = 0.0$, $\lambda_5 = 0.166666666$. The solution of the problem is :

$$\lambda_2 v_2 + \lambda_3 v_3 + \lambda_5 v_5 = 0.16666666 \begin{pmatrix} 4\\ 6\\ 6\end{pmatrix} + 0.66666666 \begin{pmatrix} 0\\ 6\\ 1\end{pmatrix} + 0.166666666 \begin{pmatrix} 0\\ 6\\ 6\end{pmatrix} =$$

$$\begin{pmatrix} 0.66666666\\ 0,999999996\\ 0,999999996\end{pmatrix} + \begin{pmatrix} 0\\ 3,99999996\\ 0.66666666\end{pmatrix} + \begin{pmatrix} 0\\ 0,999999996\\ 0,999999996\end{pmatrix} = \begin{pmatrix} 0.66666666\\ 6\\ 2,66666666\end{pmatrix} = \begin{pmatrix} x_1\\ x_2\\ x_3\end{pmatrix}$$

with objective function value of 28.