Short Course on Column Generation - Part I
Example Simplex Method

Ana Flávia U. S. Macambira
ana.macambira@academico.ufpb.br

Alain Faye
alain.faye@ensiie.fr

Departamento de Estatística
Universidade Federal da Paraíba
École nationale supérieure d'informatique pour l'industrie et l'entreprise

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Simplex Method - Example

Consider the problem below written in standard form:

$$
\begin{aligned}
\text{Maximize } z &= 5x_1 + 2x_2 \\
\text{Subject to: } x_1 + x_2 + x_3 &= 4 \\
x_1 + x_4 &= 3 \\
x_2 + x_5 &= 2 \\
x_1, x_2, x_3, x_4, x_5 &\geq 0
\end{aligned}
$$

We have matrix A:

$$
A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}
$$

which is the matrix of the coefficients of the constraints, $A \in \mathbb{R}^{3\times5}$ and its columns are:

$$
a_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, a_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, a_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, a_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, a_5 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix};
$$

Simplex Method - Example

$$\begin{aligned} \text{Maximize } z &= 5x_1 + 2x_2 \\ \text{Subject to: } x_1 + x_2 + x_3 &= 4 \\ x_1 + x_4 &= 3 \\ x_2 + x_5 &= 2 \\ x_1, x_2, x_3, x_4, x_5 &\geq 0 \end{aligned}$$

$$b = \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} \text{ is the right side of the constraints,} b \in \mathbb{R}^{3\times 1};$$

$$c = \begin{pmatrix} 5 \\ 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ is the matrix of the coefficients of the objective function, c } \in \mathbb{R}^{5\times 1}$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \text{ is the matrix of the variables,} x \in \mathbb{R}^{5\times 1}.$$

Simplex Method - Example

We split matrix A in $A = (B|N)$. We will consider

$$B = [a_3\ a_4\ a_5] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ B \in \mathbb{R}^{3\times 3} \text{ and } N = [a_1\ a_2] = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix},\ N \in \mathbb{R}^{3\times 2}.$$

$$x_B = \begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix},\quad x_N = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

$$c_B = \begin{pmatrix} c_3 \\ c_4 \\ c_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},\ c_N = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

Simplex Method - Example

$$Bx_b + Nx_N = b$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}$$

As B is the identity matrix, its inverse $B^{-1} = B$ and making $x_N = 0$, we have

$$x_B = B^{-1}b - B^{-1}Nx_N \rightarrow \text{ making } x_N = 0 \rightarrow \overline{x}_B = B^{-1}b$$

$$\begin{pmatrix} \overline{x}_3 \\ \overline{x}_4 \\ \overline{x}_5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}$$

And the current value of the objective function is:

$$\overline{z} = c_B^t \overline{x}_B = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} = 0$$

Simplex Method - Example

Once we found a basic feasible solution, we need to know if the current solution is optimal, and to answer this question we need to check if at least one non basic variable gives a positive gain to the objective function.

$$z = c_B x_B + c_N x_N$$

$$z = c_B(B^{-1}b - B^{-1}Nx_N) + c_N x_N$$

$$z = c_B B^{-1}b - c_B B^{-1}Nx_N + c_N x_N$$

$$z = \underbrace{c_B B^{-1}b}_{\bar{z}} + \underbrace{(c_N - c_B B^{-1}N)}_{\text{gain}} x_N$$

where $\bar{z}$ is the value of the current solution.

Simplex Method - Example

Now we have to see if there is at least one variable to enter the basis, so we have to calculate $c_1 - z_1$ and $c_2 - z_2$ to see if at least one non basic variable are going to improve the value of the objective function, which we are calling gain.

$$c_1 - z_1 = c_1 - c_B^\top B^{-1} a_1 = 5 - \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 5$$

$$c_2 - z_2 = c_2 - c_B^\top B^{-1} a_2 = 2 - \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 2$$

As the two non basic variables are providing some gain, we are going to choose x_1 to enter the basis because it's the biggest value.

Simplex Method - Example

- Now we know that variable x_1 is going to enter the basis;
- The question is: which variable will leave the basis?
- We have to check which variable can assume the biggest value considering that the basic variables cannot assume values less than zero.
- $x_B = \overline{x}_B - B^{-1}Nx_N$. As x_1 is going to enter the basis, $x_B = \overline{x}_B - B^{-1}a_1x_1$

Simplex Method - Example

$$x_B = \overline{x}_B - B^{-1} a_1 x_1$$

$$\begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} x_1$$

$$\begin{cases} x_3 = 4 - x_1 \\ x_4 = 3 - x_1 \\ x_5 = 2 \end{cases}$$

- As all the variables in the model need to have values greater or equal zero, x_3, x_4, x_5 need to be greater or equal zero;
- We are going to analyse to which value x_1 can increase its value respecting this condition;
- Analysing the first equation, $x_3 \geq 0$ so, we need $4 - x_1 \geq 0$, leading to $x_1 \leq 4$;
- In the second equation, $x_4 \geq 0 \rightarrow 3 - x_1 \geq 0 \rightarrow x_1 \leq 3$;
- In the third equation $x_5 \geq 0 \rightarrow 2 \geq 0$ in this case, the value of x_5 doesn't depend of the value of x_1;
- Hence, the variable that most limits the value of x_1 is x_4, so it leaves the basis.

Simplex Method - Example

Remembering the problem

$$\begin{aligned} \text{Maximize } z &= 5x_1 + 2x_2 \\ \text{Subject to: } x_1 + x_2 + x_3 &= 4 \\ x_1 + x_4 &= 3 \\ x_2 + x_5 &= 2 \\ x_1, x_2, x_3, x_4, x_5 &\geq 0 \end{aligned}$$

The first basis was

$$B = (a_3 a_4 a_5) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Now, variable x_1 entered the basis and variable x_4 has just left the basis, so our current basis is:

$$B = (a_3 a_1 a_5) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow B^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Simplex Method - Example

Now we are going to calculate the new basic feasible solution.

$$\bar{x}_B = B^{-1}b \Rightarrow \begin{pmatrix} \bar{x}_3 \\ \bar{x}_1 \\ \bar{x}_5 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$$

The value of the objective function is:

$$\bar{z} = c_B \bar{x}_B = \begin{pmatrix} 0 & 5 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = 15$$

The next step is to check if this solution is optimal by calculating the gain of the non-basic variables.

Simplex Method - Example

$$c_2 - z_2 = c_2 - c_B^\top B^{-1} a_2 = 2 - \begin{pmatrix} 0 & 5 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 2$$

$$c_4 - z_4 = c_4 - c_B^\top B^{-1} a_4 = 0 - \begin{pmatrix} 0 & 5 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = -5$$

The variable x_2 is the only one that provides a gain to the objective function, so x_2 enters the basis.

Simplex Method - Example

At the previous iteration, we decided which variable was going to leave the basis in an intuitive way. The mathematical criteria for the variable to leave the basis is:

$$x_{B_r} = \min\left\{ \frac{\bar{x}_r}{y_{rk}},\ r = 1, 2, ..., m,\ y_{rk} > 0 \right\}$$

$$y_k = B^{-1} a_k$$

At this iteration our a_k is a_2, then:

$$B^{-1}a_k = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$x_{B_r} = \min\left\{\frac{1}{1}, \frac{2}{1}\right\} = 1$, which is related to variable x_3. The variable x_2 entered the basis and the variable x_3 has just left the basis, so our current basis is:

$$B = (a_2 a_1 a_5) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \rightarrow B^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}.$$

Simplex Method - Example

Now we are going to calculate the new basic feasible solution.

$$\bar{x}_B = B^{-1}b \Rightarrow \begin{pmatrix} \bar{x}_2 \\ \bar{x}_1 \\ \bar{x}_5 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$$

The value of the objective function is::

$$\bar{z} = c_B \bar{x}_B = \begin{pmatrix} 2 & 5 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} = 17$$

The next step is to check if this solution is optimal by calculating the gain of the non-basic variables.

Simplex Method - Example

$$c_3 - z_3 = c_3 - c_B B^{-1} a_3 = 0 - \begin{pmatrix} 2 & 5 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 3 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = -2$$

$$c_4 - z_4 = c_4 - c_B B^{-1} a_4 = 0 - \begin{pmatrix} 2 & 5 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 3 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = -3.$$

As neither of the non basic variables provides a positive gain for the objective function, the current solution is optimal.