# AN EXTENSION OF THE POSSIBILITY THEORY IN VIEW OF THE FORMALIZATION OF APPROXIMATE REASONING

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**Abstract**

Our aim is here to give some indications about the basis of a logic system in order to generalize the possibility theory in the following way :

We want that each sentence receive a couple of confidence values. We call them degrees of obligation and eventuality, in order to translate the notion of truth by the half-sum of these values, and a notion that we can see as the imprecision given to this truth value : the diference eventuality - obligation which we shall call the unknown rather than uncertainty.

In relation to the possibility theory, we whish to leave the general property max(ps(P), ps(¬P)) = 1 and then leave :

$$nc(P) > 0 \Rightarrow ps(P) = 1 \quad \text{and} \quad ps(P) < 1 \Rightarrow nc(P) = 0$$

The aim of this extension is sitted in the tentation of modelising the degrees of truth and unknown, and their propagation when inferences, in the computing system about approximate reasoning.

We would like that linguistics appreciations with vagueness such "perhaps true", "probably false", "true always three times on four", "often true but very uncertain" etc..., could receive those two degrees in order to determinate an interval whose the widness would be the measure of the imprecision given to the truth degree, and it is this that we call unknown.

**Definitions**

Let us consider the algebric structure constructed by the set of couples (x,y) in $[0,1]^2$, which satisfy $x \leq y$ (hachured area on the shema),

(x will be the degree of obligation and y the degree of eventuality)

with the three internal operations :

$$\neg(x,y) = (1-y\ ,\ 1-x)$$
$$(x,y)\wedge(x',y') = (\min(x,x')\ ,\ \min(y,y'))$$
$$(x,y)\vee(x',y') = (\max(x,x')\ ,\ \max(y,y'))$$

that we shall call respectively negation, conjonction, disjonction.

Taking the definition v(x, y) = (x+y)/2, a truth degree inspirated by [Gaines 76], and i(x,y) = y-x, we shall note on the other side that four elements have a particular function, we call them :

F = (0, 0) the false, which is the minimal element for the partial ordering ≤,
T = (1, 1) the true, which is the maximum,
I = (0, 1) the uncertain, (maximum of the no-obligatory and minimum of the eventual)
and at last M = (0.5 0.5) the "certainly half true".

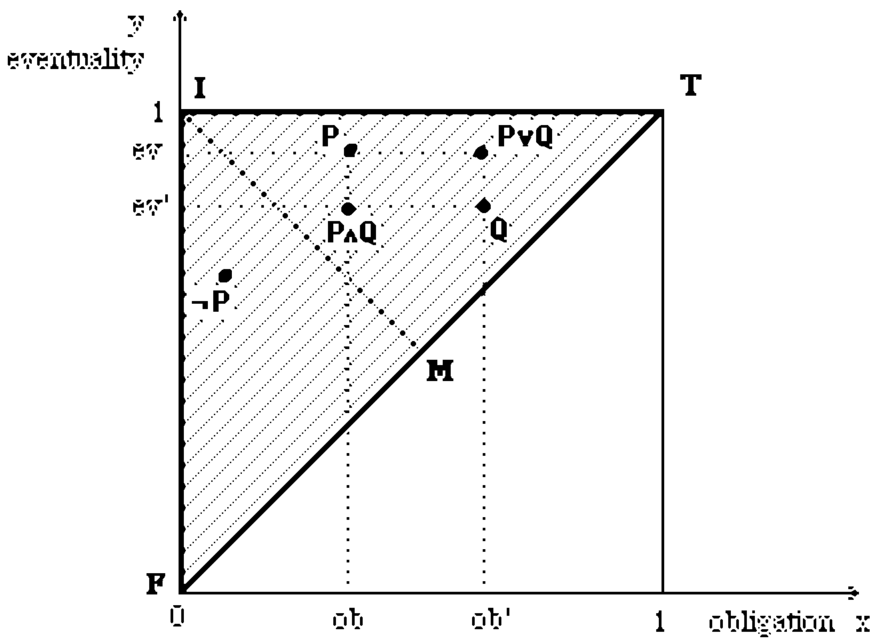
We can then call (really) eventuals couples, those of the segment IT, no-obligatory couples, those of the segment FI.

The first advantage of this axiomatic is to meet again the theory of possibilities on the cut line [FI] ∪ [IT], which has stability under the three operations ¬, ∧, ∨ above.

We have on the other side, an extension of the Kleene multivalued logic min-max (the certains couples), on the segment [FT], and at last the elementary boolean logic being only constitued by the two points F and T.

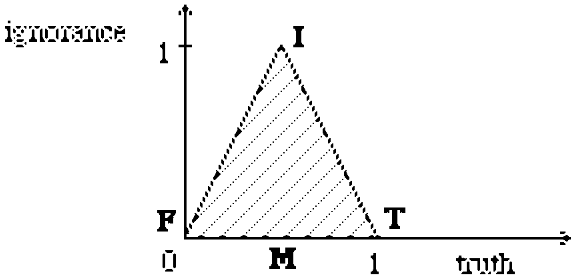
The transvers parallels lines with IM are the "equal truth lines".

If P and Q represent two couples of confidence (ob, ev) and (ob', ev'), the shema give a view of ¬P, P ∧ Q, P ∨ Q.



**Remark**

We could, in the same way, begin from the axiomatic of the couples (v, i) in $[0,1]^2$, such that : $0 \leq v - i/2 \leq v + i/2 \leq 1$ then represented by the figure :
( v = truth, i = unknown )




(with the same constants F, T, I, the equal truth lines are here the "verticals lines") the disadvantage of this second way is not having easy formulas for the definitions of $\wedge$ and $\vee$, it is the reason why we shall compute rather with the couples (ob, ev).

**Properties**

The negation is involutive (we check easily that it is a geometric symetrie in relation to IM on the two figures). We shall note that the couples on IM are equal to their negation, it will be called "half true".

The conjonction and the disjonction are idempotentes, commutatives, associatives, mutually distributives because min and max have those properties.

The Morgan properties are satistied.

We have therefore a distributive lattice, but is not a Boole algebra.

The relation $(x,y) \leq (x',y')$ defined by ($x \leq x'$ and $y \leq y'$) is a dense partial ordering having two extremities F and T.

**Implication**

We define as in classical logic the operation $\rightarrow$ by :

$$(x,y) \rightarrow (x',y') = \neg(x,y) \vee (x',y') = (\max(1-y, x'), \max(1-x, y'))$$

This definition assuming therefore the contraposition :

$$[\neg(x',y') \rightarrow \neg(x,y)] = [(x,y) \rightarrow (x',y')]$$

On [F,V], that is to say in the cases of certainty where i = 0, we then have again the implication of Kleene-Dienes : $p \rightarrow q = \max(1-p, q)$

**The question of the inference**

We could state it in the following algebrics terms : P = (ob, ev) is given and also R = P→Q = (s,t), it must resolve Q = (x,y) by the equations

$$s = \max(1-ev, x) \quad \text{and} \quad t = \max(1-ob, y)$$

a) if $\neg P \leq (s,t)$ is false, there is no solutions, concretely this could say that ¬P is "too strong" that is to say P is not true enough. It must therefore que $\neg P \leq (s,t)$.

b) if t = 1-ob and s > 1-ev, then Q is given with imprecision by (s,[0,t])

c) if s = 1-ev and t > 1-ob, then Q is given by ([0,s],t)

d) if 1-ob < t and 1-ev < s, then Q = (s,t)

A question is then raised :

Is it possible to admit the solution Q = (s,t) subject to $\neg P \leq (s,t)$ ? This solution, coming from the logic approach, is not at all acceptable in the view of a system of deduction, because, if P is known with the low values (0.1 0.2) and if the implication P→Q is quite sure with the values (0.8 0.9), then this condition would be satisfied, and we could "deduce" of it Q with those same values (0.8 0.9).

Our problem is not therefore to go on according to the formulas of the implication, but to define and to legitimate formulas about propagation of the ignorance and about the truth when activating rules. (The functional approach with $v(q) \geq v(p \wedge q)$ means that $p \wedge q$ is true as soon as p is true because q is deduced from p [Dubois - Prade 85]).

For the calculus [P = (ob, ev) , R = P→Q = (s,t)] ⇒ Q = (x,y), we must determine two functions :

x = f1(ob, ev, s, t) and y = f2(ob, ev, s, t) increasing for their 3° and 4° arguments, f1 being increasing face to face to its first argument and f2 to its second.

On the other side intuitively, it will be necessary to have Q = (ob, ev) in the best case that is to say with s = t = 1, and Q = (0, 0) in the worst of the case (s = t = 0), and more crudely P weak ⇒ Q weak, P strong ⇒ Q strong. It doesnt apply at all to what we have said previously, however, from the functions satisfiying thoses conditions we find thoses of the conjonction, meaning $Q = P \wedge (P \rightarrow Q) = P \wedge (\neg P \vee Q) = P \wedge Q$, the monotonicity $Q \leq P$ is then sure.

We can therefore choose x = s.ob and y = t.ev, or also x = min(ob, s) and y = min(ev, t), these just mentionned we shall choose because we have defined the conjonction in that way (Mamdani implication).

**Fuzzy expert system**

Let a set {p0, p1, p2, ....} named set of propositionnals variables, we define a "fuzzy expert system" by the datas of a finite number of "facts" or "litterals", that is propositionnals variables allocated with a couple (ob, ev) for each, and a finite number of "rules", that is to say implications of the type $pi_1 \wedge pi_2 \wedge pi_3 \wedge ....... \wedge pi_n \rightarrow pj$ affected with a couple (s,t).

Remark : instead of appreciate the couple (s, t) de P → Q, some systems use the couple formed by s = nc(P → Q), n = nc(Q → P) [Martin-Clouaire 84].

The theory developped by a such system is by definition the smallest set of facts (propositionnals variables with couples of confidence), contening the initials facts, and closed by deduction. Then the questions raised are : what happens in case of normalization of two couples of confidence infered to a same propositionnal value ? and is finite the theory (stop the inferences ?)

**The normalization**

Let us take the case of $P_0$ = (ob, ev) , R = P→Q = (s,t) driving to Q = (min(ob, s), min(ev, t))

next, from Q = (min(ob,s), min(ev,t)) , R' = Q→P = (u,v) which bring on this time :

$P_1$ = (min(ob,s,u), min(ev,t,v)) with $P_1 \leq P_0$ that is seems absolutely normal, we can advise that there is no inference if the result is less than the one which is already known for a conclusion [Gacôgne 90].

But, more generally if two rules suceed to the same conclusion :

$R_1 = P_1 \rightarrow Q = (s_1,t_1)$ and $R_2 = P_2 \rightarrow Q = (s_2,t_2)$, it is necessary to adopt a principe of normalization for the conclusion, which is not depending of the rule ordering, this one will be realised by taking the max of the confidences brought to the conclusion by the differents rules.

Let us say that in the FRIL system [Baldwin 90], a support (n,p) is affected to each fact, and if a fact is known by several proofs concluing to the supports (n1,p1) and (n2,p2), then it is the operation (max(n1,n2), min(p1,p2)) which is choosed subject to $n \leq p$.

We shall adopt further the following approach :

If a conclusion Q is infered with the values (obi, evi) directely or by a contraposition.

a) If neither Q nor ¬Q are known that is to say : not in the base of the facts, then (Q obi evi) is of course added in this base.

b) If Q is known with the values (oba, eva), then it is replaced by Q with the new values max(obi, oba), max(evi, eva))

c) If ¬Q is known with the values (obn, evn), then it seems natural to think that the knowledge about ¬Q present more interest than it of its opposite, and in this case it is ¬Q which is replaced by (¬Q, max(obn, 1-evi) max(evn, 1-obi))

In all cases, an already known fact will be therefore modified by an inference only if its confidence grows, that means eventually the confidence in its opposite grows down.

**Example**

Let us consider the sentences : P = "It rains", M = "I am damped", S = "I go out", A = "I have an umbrella", R = "I take an umbrella".

We give the five rules given with their conclusion and next their premisses (the negations are in parenthesis) :

```
(set 'BR '  ((6 8 (S) P)         ;if it rains, i don't go out
             (6 10 R A)          ;if i have an umbrella, i take it
             (8 10 R S P)        ;if i go out and it rains, i take my umbrella
             (7 10 (M) R)        ;i have an umbrella, then i'm not damped
             (9 10 (S) P M)))    ;it rains and i'm damped, i stay in
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In order to simplify the visualisation, the coefficients are given in tenth, a short Lisp program whose the top function "sef", give its results.

(set 'LF '( (P 4 8) (S 7 7) )); are the two initials facts given with their two values, the trace of reasonning is the next :

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(sef LF BR)
; rule : ( 6  8 (s) p) with : ( 4 8)
; opposite of the conclusion modified
; rule : ( 8 10 r s p) with : ( 4  7)
; unknown conclusion, therefore accepted
; rule : (0  4 (a) (r)) with : ( 3 6)
; unknown conclusion, therefore accepted
; rule : ( 6 10 r a) with : ( 6 10)
; conclusion modified
; rule : ( 7 10 (m) r) with : ( 6 10)
; unknown conclusion, therefore accepted
; rule : ( 9 10 (s) p m) with : (0  4)
; opposite of the conclusion modified
= ((p  4  8) ((a) 0  4) (r  6 10) ((m) 6 10) (s  7 10))
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We can remark, in the final base of facts, that the fact S is stronger in relation to its initials values.

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