Linking Focusing and Resolution with Selection

Guillaume Burel

ENSIIE/Samovar
guillaume.burel@ensiie.fr

Abstract. Focusing and selection are techniques that shrink the proof search space for respectively sequent calculi and resolution. To bring out a link between them, we generalize them both: we introduce a sequent calculus where each *occurrence* of an atom can have a positive or a negative polarity; and a resolution method where each literal, whatever its sign, can be selected. We prove the equivalence between cut-free proofs in this sequent calculus and derivations of the empty clause in that resolution method. Such a generalization is naturally not semi-complete in general; we present three complete instances: first, our framework allows us to show that usual focusing corresponds to hyperresolution and semantic resolution; the second instance is deduction modulo theory; and a new setting extends deduction modulo theory with rewriting rules having several left-hand sides, therefore restricting even more the proof search space.

Keywords: proof theory, sequent calculus, refinements of resolution, deduction modulo theory, polarization

Introduction

In addition to clever implementation techniques and data structures, a key point that explain the success of state-of-the-art automated theorem provers is the use of calculi that dramatically reduce proof search space. In the last decades, one can highlight the independent developments of two families of techniques. First, in the kind of methods based on resolution, proof search space can be shrunk using ordering and selection techniques. The intuition is to restrict the application of the resolution rule to only some literals in a clause. If equality is considered, this leads to the superposition calculus which is the base calculus of the most efficient current automated provers for first-order classical logic. Second, in sequent calculi, Andreoli [1] introduced a technique called focusing to reduce non-determinism in the application of sequent-calculus rules first by applying all invertible rules and second by chaining the application of non-invertible rules. Originally developed for linear logic, focusing has been extended to intuitionistic and classical first-order logic [25]. Focusing is mostly used in fields where sequent calculi, and related inverse and tableaux methods, are the most accurate proving method. For instance, there exists tools for first-order linear logic [12], for intuitionistic logic [26] and for modal logic [27]. Focusing is also the key ingredient in Miller's ProofCert project for a universal framework for proof certification.

Despite their apparent lack of relation, we show in this paper that selection in refinements of the resolution calculus and focusing in sequent calculus are in fact strongly related, so that usual focusing in classical first-order logic corresponds in fact to negative hyperresolution, where all negative literals are selected in a clause and are resolved at once. This connection is obtained by slightly generalizing both techniques: concerning resolution, we allow any literal of the input clauses to be selected, whatever its sign; for the focusing part, we allow polarization not only of connectives, but also of all occurrences of literals. The main theorem of this paper, Theorem 1, shows that the sets of clauses whose insatisfiability can be proved by the resolution method with generalized selection are the sequents that have a cut-free proof in the generalized focusing setting.

This generalization allows us to cover of wider spectrum of proof systems. In particular, this permits to consider systems that search for proofs modulo some theory. Indeed, in real world applications, proof obligations are often verified within one or several theories. This explains the interest in and the success of Sat Modulo Theory tools in the recent years. We would like to have a system that is generic enough to cover all kind of theories (arithmetics, higher order, etc.) We also want to have efficient proof search methods associated with this framework. Let us consider a proof search method $\mathbb{P}(\mathcal{T})$ parameterized by a theory $\mathcal{T}$. We say that $\mathbb{P}(\mathcal{T})$ is relatively consistent if, whatever its paramater $\mathcal{T}$, it terminates with the answer "NO" when given the false formula $\perp$ as input. In other words, if the theory is consistent, the method will not try to prove the trivially false formula. As pointed out in [9], we cannot have a generic proof that would work for all $\mathcal{T}$ of the completeness of a relatively consistent method $\mathbb{P}(\mathcal{T})$. Indeed, such a proof would imply the consistency of the theory $\mathcal{T}$, and according to Gödel cannot be performed in $\mathcal{T}$ itself. So to get a sufficiently general framework, we cannot hope to have a proof method whose completeness is proved once for all. Indeed, our generalizations of resolution with selection and of focusing are not complete in all cases. However, we prove that both of them represent the same fragment.

We then give three instances of our framework, where we can have proofs of completeness. First, as stated above, we link usual focusing with hyperresolution, and, in the ground case, with semantic resolution. Second, we show that Deduction Modulo Theory [18] is also a particular instance of this framework. Third, we give an even more restricted calculus, and we show how its completeness can be reduced to completeness of several instances of Deduction Modulo Theory. To demonstrate this last part, let us consider by example the theory defining the powerset:

$$\forall X,\ \forall Y,\ X \in \mathcal{P}(Y) \Leftrightarrow (\forall Z,\ Z \in X \Rightarrow Z \in Y)$$

This theory can be put in clausal normal form, using d as a Skolem symbol, and we select (by underlining them) some literals in these clauses:

$$\underline{\neg X \in \mathcal{P}(Y)} \cup \underline{\neg Z \in X} \cup Z \in Y \tag{1}$$

$$\underline{X \in \mathcal{P}(Y)} \cup d(X,Y) \in X \tag{2}$$

$$\underline{X \in \mathcal{P}(Y)} \cup \underline{\neg d(X,Y) \in Y} \tag{3}$$

In our framework, this would lead to the following synthetic rules, that can be used in place of the axioms:

$$\text{(1)}\vdash \; \dfrac{\Delta, u \in \mathcal{P}(v), t \in u, t \in v \vdash}{\Delta, u \in \mathcal{P}(v), t \in u \vdash} \qquad\qquad \text{(2)}\vdash \; \dfrac{}{\Delta, \neg u \in \mathcal{P}(v), d(u,v) \in v \vdash}$$

$$\text{(3)}\vdash \; \dfrac{\Delta, \neg u \in \mathcal{P}(v), d(u,v) \in u \vdash}{\Delta, \neg u \in \mathcal{P}(v) \vdash}$$

The only proof of transitivity of the membership in the powerset is then

$$\exists\vdash\;\dfrac{\wedge\vdash\;\dfrac{\text{(3)}\vdash\;\dfrac{\text{(1)}\vdash\;\dfrac{\text{(1)}\vdash\;\dfrac{\text{(2)}\vdash\;\dfrac{}{a \in \mathcal{P}(b), b \in \mathcal{P}(c), \neg a \in \mathcal{P}(c), d(a,c) \in a, d(a,c) \in b, d(a,c) \in c \vdash}}{a \in \mathcal{P}(b), b \in \mathcal{P}(c), \neg a \in \mathcal{P}(c), d(a,c) \in a, d(a,c) \in b \vdash}}{a \in \mathcal{P}(b), b \in \mathcal{P}(c), \neg a \in \mathcal{P}(c), d(a,c) \in a \vdash}}{a \in \mathcal{P}(b), b \in \mathcal{P}(c), \neg a \in \mathcal{P}(c) \vdash}}{a \in \mathcal{P}(b) \wedge b \in \mathcal{P}(c) \wedge \neg a \in \mathcal{P}(c) \vdash}}{\exists A.\ \exists B.\ \exists C.\ A \in \mathcal{P}(B) \wedge B \in \mathcal{P}(C) \wedge \neg A \in \mathcal{P}(C) \vdash}$$

where the active formulas in a sequent are underwaved, and double lines indicates potentially several applications of an inference rule.

On the resolution side, clauses (1) to (3) leads to the following ground derived rules:

$$\text{(1)}\;\dfrac{u \in \mathcal{P}(v) \cup C \qquad t \in u \cup D}{t \in v \cup C \cup D} \qquad\qquad \text{(2)}\;\dfrac{\neg u \in \mathcal{P}(v) \cup C}{d(u,v) \in u \cup C}$$

$$\text{(3)}\;\dfrac{\neg u \in \mathcal{P}(v) \cup C \qquad d(u,v) \in v \cup D}{C \cup D}$$

Once again, there is only one proof of transitivity, i.e. starting from the set of clauses $\{a \in \mathcal{P}(b); b \in \mathcal{P}(c); \neg a \in \mathcal{P}(c)\}$:

$$\text{(3)}\;\dfrac{\neg a \in \mathcal{P}(c) \qquad \text{(1)}\;\dfrac{b \in \mathcal{P}(c) \qquad \text{(1)}\;\dfrac{a \in \mathcal{P}(b) \qquad \text{(2)}\;\dfrac{\neg a \in \mathcal{P}(c)}{d(a,c) \in a}}{d(a,c) \in b}}{d(a,c) \in c}}{\emptyset}$$

and we cannot even infer other clauses than those. We let the reader compare with what happens if we used clauses (1) to (3) in resolution, even using the ordered resolution with selection refinement.

Related work. As we do here, Goubault-Larrecq [22] proves completeness of several refinement of resolution, including hyperresolution and semantic resolution, using syntactic transformations instead of relying on the construction of a model.

Farooque, Graham-Lengrand, and Mahboubi [21] developed a sequent calculus, based on focusing, that is able to simulate DPLL($\mathcal{T}$), the most common calculus used in SMT provers. The main difference with our framework is that in [21], the theory is considered as a black box which is called as an oracle. Here, the theory is considered as a first-class citizen.

Within the ProofCert project, the tool `Checkers` of Chihani, Libal, and Reis [13] is able to verify proofs coming from automated theorem provers based on resolution such as E-prover, using a focused sequent calculus as the kernel of the checker.

Notations and conventions. We use standard definitions for terms, predicates, propositions (with connectives $\bot, \top, \neg, \wedge, \vee$ and quantifiers $\forall, \exists$), sequents and substitutions. A literal is an atom or its negation. A clause is a set of literals. We will identify a literal with the singleton clause containing it. Unless stated otherwise, letters $P, Q, R, P', P_1, \dots$ denote atoms, $L, K, L', L_1, \dots$ denote literals, $C, D, C', C_1, \dots$ denote clauses, Γ, Δ denote set of clauses or set of formulas (depending on the context). $A^\perp$ denotes the negative normal form of $\neg A$.

1 Focusing with Polarized Occurrences of Atoms

Focusing was introduced by Andreoli [1] to restrict the non-determinism in the sequent calculus for linear logic. It relies on the alternation of two phases: During the asynchronous phase (sequents with $\Uparrow$), all invertible rules are applied on the formulas of the sequent. During the synchrounous phase (sequents with $\Downarrow$), a particular formula is selected —the focus is on it— and all possible non-invertible rules are successively applied on it. This idea has been extended by Liang and Miller [25] to intuitionistic and classical first-order logic. In these, connectives may have invertible and non-invertible version of their sequent calculus rules. Therefore, one considers in that case two versions of the connective, one positive for the non-invertible case, and one negative for the invertible case. Some connectives, i.e. $\exists$ in classical logic, only have a positive version, and dually, others, such has $\forall$ in classical logic, only have a negative version. Given an usual formula, one can decide which version of a connective one wants to use at a particular occurrence, which is called a polarization of the formula. Similarly, one can decide a polarity for each atoms A, with $\neg A$ having the inverse polarity. If a positive literal L is focused on in a branch, then necessarily this branch must be closed, with $L^\perp$ in the same context. (See rule $\widehat{\Downarrow\vdash}$ in Figure 1.) However, this polarity is usually chosen globally for each occurrences of the atom. Besides, note that to inverse the polarity of a formula, e.g. to impose a change of phase, one can prefix it by so-called delays: $\delta^- A$ is negative whatever the polarity of A. Delays can be defined for instance by $\delta^- A = \forall x.\ A$ where x is not free in A, so we do not need them in the syntax and the rules.

Liang and Miller [25] introduced the sequent calculus LKF, and proved it to be complete for classical first-order logic. In Figure 1, we present the calculus LKF$^\perp$, which is almost the same with the following differences:

$$\widehat{\Uparrow}\vdash\ \frac{}{\Gamma, L, L^{\perp} \Uparrow \vdash} \qquad \widehat{\Downarrow}\vdash\ \frac{}{\Gamma, L^{\perp} \Downarrow \underline{L} \vdash}$$

$$\Uparrow\exists\vdash\ \frac{\Gamma \Uparrow \Delta, A \vdash}{\Gamma \Uparrow \Delta, \exists x.\ A \vdash}\ x \text{ not free in } \Gamma, \Delta \qquad \Downarrow\forall\vdash\ \frac{\Gamma \Downarrow \{t/x\}A \vdash}{\Gamma \Downarrow \forall x.\ A \vdash}$$

$$\Uparrow\vee\vdash\ \frac{\Gamma \Uparrow \Delta, A \vdash \qquad \Gamma \Uparrow \Delta, B \vdash}{\Gamma \Uparrow \Delta, A \vee^{-} B \vdash} \qquad \Downarrow\vee\vdash\ \frac{\Gamma \Downarrow A \vdash \qquad \Gamma \Downarrow B \vdash}{\Gamma \Downarrow A \vee^{+} B \vdash}$$

$$\Uparrow\wedge\vdash\ \frac{\Gamma \Uparrow \Delta, A, B \vdash}{\Gamma \Uparrow \Delta, A \wedge^{-} B \vdash} \qquad \Downarrow\wedge_1\vdash\ \frac{\Gamma \Downarrow A \vdash}{\Gamma \Downarrow A \wedge^{+} B \vdash}$$

$$\Uparrow\top\vdash\ \frac{\Gamma \Uparrow \Delta \vdash}{\Gamma \Uparrow \Delta, \top \vdash} \qquad \Downarrow\wedge_2\vdash\ \frac{\Gamma \Downarrow B \vdash}{\Gamma \Downarrow A \wedge^{+} B \vdash}$$

$$\text{Focus}\ \frac{\Gamma, A \Downarrow A \vdash}{\Gamma, A \Uparrow \vdash}\ A \text{ positive} \qquad \Downarrow\bot\vdash\ \frac{}{\Gamma \Downarrow \bot \vdash}$$

$$\overset{\Uparrow}{\leftarrow}\ \frac{\Gamma, A \Uparrow \Delta \vdash}{\Gamma \Uparrow A, \Delta \vdash}\ A \text{ positive or literal} \qquad \text{Release}\ \frac{\Gamma \Uparrow A \vdash}{\Gamma \Downarrow A \vdash}\ A \text{ negative}$$

Fig. 1. The sequent calculus LKF$^{\perp}$

– All formulas are put on the left-hand side of the sequent, instead of the right-hand side. Therefore, one does not try to prove a disjonction of formulas, but one tries to refute a conjonction of formulas. This is the same thanks to the dual nature of classical first-order logic, and this helps to be more close from the resolution derivations. Note that consequently, $\exists$ becomes negative and $\forall$ positive.
– The polarity of an atom is not chosen globally, but each *occurrence* of an atom can have a positive or a negative polarity. In particular, we can have two literals L and $L^{\perp}$ which are both negative, or both positive. We denote by $\underline{L}$ the fact that the literal L has a positive polarity. To be able to close branches on which we have two negative opposed literals, we added a rule $\widehat{\Uparrow}\vdash$.

We denote by $\Gamma \Uparrow\vdash$ the fact that there exist a proof of the sequent $\Gamma \Uparrow\vdash$ in LKF$^{\perp}$.

LKF$^{\perp}$ is not complete in general. One of the simplest example of incompleteness is the sequent $\underline{P} \vee^{+} Q, \underline{\neg P} \vee^{+} Q, \neg Q \Uparrow\vdash$ which has no proof although $P \vee Q, \neg P \vee Q, \neg Q$ is not satisfiable.

2 Resolution with Input Selection

Ordered Resolution with Selection was introduced by Bachmair and Ganzinger [2] (see also Bachmair and Ganzinger [3]) as a complete refinement of resolution. It is parameterized by a Noetherian ordering $\succ$ and a selection function $\mathcal{S}$. If the selection function selects at least a literal, only those can be used in Resolution. Otherwise, only the maximal literals w.r.t. $\succ$ can be used. In the usual setting, the selection function can only select a subset of the *negative* literals of a clause. Here, we relax this condition : any literal can be selected, however only in the

$$\textsf{Resolution with Selection}\ \frac{\underline{K_1} \cup \ldots \cup \underline{K_n} \cup C \qquad {K'_1}^{\perp} \cup D_1 \qquad \ldots \qquad {K'_n}^{\perp} \cup D_n}{\sigma(C \cup D_1 \cup \ldots \cup D_n)}$$

- $\mathcal{S}(\underline{K_1} \cup \ldots \cup \underline{K_n} \cup C) = \{K_1; \ldots; K_n\}$
- $\mathcal{S}({K'_i}^{\perp} \cup D_i) = \emptyset$
- σ is the most general unifier of the simultaneous unification problem $K_1 =^? K'_1, \ldots, K_n =^? K'_n$

$$\textsf{Resolution}\ \frac{L \cup C \qquad {L'}^{\perp} \cup D}{\sigma(C \cup D)} \qquad\qquad \textsf{Factoring}\ \frac{L \cup L' \cup C}{\sigma(L \cup C)}$$

- $\mathcal{S}(L \cup C) = \emptyset$
- $\mathcal{S}({L'}^{\perp} \cup D) = \emptyset$
- σ is the most general unifier of $L =^? L'$

- $\mathcal{S}(L \cup L' \cup C) = \emptyset$
- σ is the most general unifier of $L =^? L'$

Fig. 2. Resolution with Input Selection

input clauses. That means that $\mathcal{S}(C) = \emptyset$ for the generated clauses. We also allow to have the same input clause several times with different selections. Of course, we lose completeness in general.

The inference rules of Resolution with Input Selection are presented in Fig. 2. Literals that are selected in a clause are underlined. We will see that they indeed correspond to the literals that have a positive polarization in LKF$^{\perp}$. As usual, variables are renamed in the clauses to avoid that premises of the inference rules share variables. Note that two clauses with a non-empty selection cannot be resolved together. This is reminiscent of the set-of-support strategy for resolution [30].

Definition 1 (Resolution derivation). *We write $\Gamma \rightsquigarrow C$ if C can be derived from some clauses in Γ using the inference rule* Resolution with Selection*,* Resolution*, or* Factoring *presented in Figure 2.*

We write $\Delta; \Gamma \rightsquigarrow^* C$ if

- *$C \in \Delta \cup \Gamma$ or if*
- *there exists D such that $\Gamma \rightsquigarrow D$ and $\Gamma, D \rightsquigarrow^* C$.*

As usual, since all rules are sound, the goal is to produce the empty clause $\emptyset$ starting from a set of clauses Γ to show that Γ is unsatisfiable. Here again, the calculus is not complete: from the set of clauses $\underline{P} \cup Q, \underline{\neg P} \cup Q, \neg Q$, no inference rule can be applied: to apply Resolution with Selection, we would need a clause where P, or $\neg P$, is not selected, and Resolution needs two clauses without selection.

3 Focusing is a Conservative Extension of Resolution with Selection

To link LKF$^{\perp}$ with Resolution with Input Selection, we need to indicate how clauses are related to polarized formulas.

Definition 2. *Given a clause $C = \{L_1; \ldots; L_n; K_1; \ldots; K_m\}$ whose free variables are $x_1, \ldots, x_l$ and such that $\mathcal{S}(C) = \{L_1; \ldots; L_n\}$, we define the associated formula $\ulcorner C \urcorner = \forall x_1, \ldots, x_l.\ \underline{L_1} \vee^+ \cdots \vee^+ \underline{L_n} \vee^+ K_1 \vee^+ \cdots \vee^+ K_m$. $\ulcorner C \urcorner$ is said to be in clausal form. By extension, $\ulcorner \Gamma \urcorner$ is the set of the formulas associated to the clauses of the set Γ.*

Theorem 1. *Let Γ be a set of clauses. We have $\ulcorner \Gamma \urcorner \Uparrow\vdash$ iff $\Gamma \leadsto^* \emptyset$.*

The two direction of the proof are given in the next sections. Due to a lack of space, some proofs are given in the appendix.

3.1 From Focused Proofs to Resolution Derivations

We need a few lemmas to prove the first direction.

Lemma 1. *For all clauses $C_1, \ldots, C_n$ and D such that $\mathcal{S}(C_i) = \emptyset$ for all i and $\mathcal{S}(D) = \emptyset$, if $\Gamma, C_1, \ldots, C_n \leadsto^* \emptyset$ and $\Gamma, D \leadsto^* \emptyset$, then $\Gamma, C_1 \cup D, \ldots, C_n \cup D \leadsto^* \emptyset$ where $\mathcal{S}(C_i \cup D) = \emptyset$ for all i.*

Corollary 1. *For all clauses $C_1, \ldots, C_n$ such that $\mathcal{S}(C_i) = \emptyset$ for all i. If $\Gamma, C_i \leadsto^* \emptyset$ for all i then $\Gamma, C_1 \cup \cdots \cup C_n \leadsto^* \emptyset$ where $\mathcal{S}(C_1 \cup \cdots \cup C_n) = \emptyset$.*

Lemma 2. *For all clauses C, for all substitutions θ, consider that $\mathcal{S}(\theta C) = \emptyset$, If $\Gamma, \theta C \leadsto^* \emptyset$ then $\Gamma, C \leadsto^* \emptyset$ where $\mathcal{S}(C) = \emptyset$.*

Theorem 2. *If $\ulcorner \Gamma \urcorner \Uparrow\vdash$, then $\Gamma \leadsto^* \emptyset$.*

Proof. By induction on the proof $\ulcorner \Gamma \urcorner \Uparrow\vdash$. Since $\ulcorner \Gamma \urcorner$ contains only formulas in clausal form, there are only four cases :

– $\widehat{\vdash}\ \dfrac{}{\Gamma', L, L^\perp \Uparrow \vdash}$
  In that case, we can simply apply Resolution on L and $L^\perp$ to derive $\emptyset$, hence $\ulcorner \Gamma \urcorner', L, L^\perp \leadsto \emptyset$.
– The proof focuses on a formula corresponding to the empty clause:
$$\text{Focus}\ \dfrac{\Downarrow\perp\vdash\ \dfrac{}{\ulcorner \Gamma \urcorner \Downarrow \perp \vdash}}{\ulcorner \Gamma \urcorner \Uparrow \vdash}$$
  In that case, $\emptyset$ already belongs to Γ.
– The proof focuses on a formula corresponding to a clause without selection. Because of focusing constraints, the proof is necessarily of the form:
$$\text{Focus}\ \dfrac{\Downarrow\forall\vdash\ \dfrac{\Downarrow\vee\vdash\ \dfrac{\text{Release}\ \dfrac{\overset{\Uparrow}{\leftarrow}\ \dfrac{\ulcorner \Gamma \urcorner, \theta L_1 \Uparrow \vdash}{\ulcorner \Gamma \urcorner \Uparrow \theta L_1 \vdash}}{\ulcorner \Gamma \urcorner \Downarrow \theta L_1 \vdash} \quad \cdots \quad \text{Release}\ \dfrac{\overset{\Uparrow}{\leftarrow}\ \dfrac{\ulcorner \Gamma \urcorner, \theta L_m \Uparrow \vdash}{\ulcorner \Gamma \urcorner \Uparrow \theta L_m \vdash}}{\ulcorner \Gamma \urcorner \Downarrow \theta L_m \vdash}}{\ulcorner \Gamma \urcorner \Downarrow \theta(L_1 \vee^+ \cdots \vee^+ L_m) \vdash}}{\ulcorner \Gamma \urcorner \Downarrow \overline{\forall x}.L_1 \vee^+ \cdots \vee^+ L_m \vdash}}{\ulcorner \Gamma \urcorner \Uparrow \vdash}$$
  By induction hypothesis, we have derivations of $\Gamma, \theta L_k \leadsto^* \emptyset$ for all $1 \leq k \leq m$. By Corollary 1, we have a derivation $\Gamma, \theta(L_1 \cup \cdots \cup L_k) \leadsto^* \emptyset$ with nothing selected in $\theta(L_1 \cup \cdots \cup L_k)$. By Lemma 2, we have a derivation of $L_1 \cup \cdots \cup L_k \leadsto^* \emptyset$.

– The proof focuses on a formula corresponding to a clause with selection. Because of focusing constraints, the proof is necessarily of the form:

$$\text{Focus}\;\dfrac{\Downarrow\forall\vdash\;\dfrac{\Downarrow\vee\vdash\;\dfrac{\cdots\;\widehat{\Downarrow}\vdash\dfrac{}{\ulcorner\Gamma\urcorner \Downarrow \theta\underline{K_j} \vdash}\quad\cdots\quad\text{Release}\;\dfrac{\overleftarrow{\Uparrow}\;\dfrac{\ulcorner\Gamma\urcorner, \theta L_k \Uparrow \vdash}{\ulcorner\Gamma\urcorner \Uparrow \theta L_k \vdash}}{\ulcorner\Gamma\urcorner \Downarrow \theta L_k \vdash}\;\cdots}{\ulcorner\Gamma\urcorner \Downarrow \theta(\underline{K_1} \vee^+ \cdots \vee^+ \underline{K_n} \vee^+ L_1 \vee^+ \cdots \vee^+ L_m) \vdash}}{\ulcorner\Gamma\urcorner \Downarrow \overline{\forall x}.\underline{K_1} \vee^+ \cdots \vee^+ \underline{K_n} \vee^+ L_1 \vee^+ \cdots \vee^+ L_m \vdash}}{\ulcorner\Gamma\urcorner \Uparrow \vdash}$$

where $\overline{\forall x}.\underline{K_1} \vee^+ \cdots \vee^+ \underline{K_n} \vee^+ L_1 \vee^+ \cdots \vee^+ L_m$ and $\theta {K_j}^\perp$ for all $1 \leq j \leq n$ are member of $\ulcorner\Gamma\urcorner$.
By induction hypothesis, we have derivations of $\Gamma, \theta L_k \rightsquigarrow^* \emptyset$ for all $1 \leq k \leq m$. By Corollary 1, we have a derivation $\Gamma, \theta(L_1 \cup \cdots \cup L_k) \rightsquigarrow^* \emptyset$.
From $\underline{K_1} \cup \cdots \cup \underline{K_n} \cup L_1 \cup \cdots \cup L_m$, $\theta {K_1}^\perp$, ..., and $\theta {K_n}^\perp$ we can apply Resolution with Selection to obtain $\theta(L_1 \cup \cdots \cup L_m)$, hence a derivation $\Gamma \rightsquigarrow^* \emptyset$. □

3.2 From Resolution Derivations to Focused Proofs

Lemma 3. *For all clauses C with $\mathcal{S}(C) = \emptyset$, for all substitution σ, if $\Gamma, \ulcorner C\urcorner, \ulcorner \sigma C\urcorner \Uparrow\vdash$, then $\Gamma, \ulcorner C\urcorner \vdash$.*

Lemma 4. *For all clauses $L \cup C$ and $L'^\perp \cup D$ without selection, if $\sigma = mgu(L, L')$, if $\Gamma, \ulcorner L \cup C\urcorner, \ulcorner L'^\perp \cup D\urcorner, \ulcorner\sigma(C \cup D)\urcorner \Uparrow\vdash$ then $\Gamma, \ulcorner L \cup C\urcorner, \ulcorner L'^\perp \cup D\urcorner \Uparrow\vdash$.*

Lemma 5. *For all clauses $\underline{K_1} \cup \ldots \cup \underline{K_n} \cup C$, $K_1'^\perp \cup D_1$, ..., and $K_n'^\perp \cup D_n$, where $\mathcal{S}(\underline{K_1} \cup \ldots \cup \underline{K_n} \cup C) = \{K_1; \ldots; K_n\}$, $\mathcal{S}(K_i'^\perp \cup D_i) = \emptyset$ and σ is the most general unifier of the simultaneous unification problem $K_1 =^? K_1', \ldots, K_n =^? K_n'$, if $\Gamma, \ulcorner \underline{K_1} \cup \ldots \cup \underline{K_n} \cup C\urcorner, \ldots \ulcorner K_i'^\perp \cup D_i\urcorner, \ldots, \ulcorner\sigma(C \cup D_1 \cup \cdots \cup D_n)\urcorner \Uparrow\vdash$ then $\Gamma, \ulcorner \underline{K_1} \cup \ldots \cup \underline{K_n} \cup C\urcorner, \ldots \ulcorner K_i'^\perp \cup D_i\urcorner, \ldots \Uparrow\vdash$.*

Theorem 3. *If $\Gamma \overset{\succ}{\rightsquigarrow}{}^* \emptyset$, then $\ulcorner\Gamma\urcorner \Uparrow\vdash$.*

Proof. By induction on the derivation $\Gamma \overset{\succ}{\rightsquigarrow}{}^* \emptyset$. If $\emptyset$ is in Γ, then we focus on $\ulcorner\emptyset\urcorner = \bot$ and apply $\Downarrow\bot\vdash$. If the first step is Factoring, we apply Lemma 3. If it is Resolution, we apply Lemma 4. If it is Resolution with Selection, we apply Lemma 5. □

4 Complete Instances

4.1 Usual Focusing and Semantic Hyperresolution

As said earlier, in LKF, not all occurrences of literals can have an arbitrary polarity. Instead, each atom P is given globally a polarity, and $P^\perp$ has the opposite polarity.

Let us first look at the simple case where atoms are given a positive polarity.

Theorem 4 (Corollary of [25, Theorem 17]). *If the literals with a positive polarity are exactly the atoms, LKF$^\perp$ is (sound and) complete.*

If we look at the corresponding resolution calculus, the Resolution with Selection becomes:

$$\text{Resolution with Selection}\ \frac{P_1 \cup \ldots \cup P_n \cup C \qquad \neg P'_1 \cup D_1 \qquad \ldots \qquad \neg P'_n \cup D_n}{\sigma(C \cup D_1 \cup \ldots \cup D_n)}$$

where C and D_i for all i contain only negative literals, and σ is the most general unifier of $P_1 =^? P'_1, \ldots, P_n =^? P'_n$. Note that the clause $\sigma(C \cup D_1 \cup \ldots \cup D_n)$ contains only negative literals, so that no literal is selected in it, as required by our calculus.

Besides, Resolution cannot be applied any longer, since there exists no clause $P \cup C$ with $\mathcal{S}(P \cup C) = \emptyset$.

The corresponding resolution calculus is in fact negative hyperresolution [28]: premises of an inference contains all only positive literals, except one clause whose all negative literals are resolved at once.

Theorem 1 therefore links usual focusing with negative hyperresolution. Of course it is also possible to obtain positive hyperresolution by requiring all atoms to have a negative polarity.

Let us now look at the general case, where atoms are given an arbitrary polarity. Let us stick to the ground case. In that case, note that the literals with a positive polarity form a Herbrand interpretation I, i.e. a model whose domain is the set of terms interpreted as themselves. We therefore have:

Theorem 5 (Corollary of [25, Theorem 17]). *Given a Herbrand interpretation I, if literals L with a positive polarity are exactly those such that $I \models L$, LKF$^\perp$ is (sound and) complete.*

If we look at the corresponding resolution calculus, the Resolution with Selection becomes:

$$\text{Resolution with Selection}\ \frac{K_1 \cup \ldots \cup K_n \cup C \qquad {K_1}^\perp \cup D_1 \qquad \ldots \qquad {K_n}^\perp \cup D_n}{C \cup D_1 \cup \ldots \cup D_n}$$

where for all i, $I \models K_i$, $I \not\models C$ and $I \not\models D_i$. Note that $I \not\models C \cup D_1 \cup \ldots \cup D_n$, so that no literal is selected in the generated clause as required by our calculus.

Besides, Resolution cannot be applied any longer, since we cannot have clauses $P \cup C$ and $\neg P \cup D$ where both P and $\neg P$ are not in I.

The corresponding resolution calculus is in fact semantic hyperresolution [29][11, Sect. 1.3.5.3]: all the premises of an inference step must be false in I, except one clause whose all literals that are true in I are resolved at once.

Theorem 1 therefore links usual focusing in the ground case with semantic resolution.

Completeness of LKF therefore leads to proofs of completeness of semantic resolution in the ground case, and hyperresolution in the first-order case, that do not rely on the construction of a model. To our knowledge, only Goubault-Larrecq [22] showed a similar result. We also provided a proof similar to the one in this paper, but restricted to Ordered Resolution, without selection of literals [6].

Note that we cannot extend this link between semantic resolution and focusing to the non-ground case because given a literal L such that $I \not\models L$, we can have an instance σL such that $I \models \sigma L$, so that σL should be selected in the generated clause, but our resolution calculus cannot take this into account.

4.2 Deduction Modulo Theory

Deduction Modulo Theory is a framework introduced by Dowek et al. [18] that consists in applying the inference rules of an existing proof system modulo some congruence over formulas. This congruence represents the theory, and it is in general defined by means of rewriting rules. To be expressive enough, these rules are defined not only at the term level, but also for formulas. To get simpler presentations of theories, we distinguish between rewrite rules that can be applied at positive and at negative positions by giving them a polarity, where by negative position we mean under a odd number of $\neg$. We therefore have positive rules $P \rightarrow^+ A$ and negative rules $P \rightarrow^- A$ where P is an atom and A an arbitrary formula whose free variables appears in P. Given a rule $P \rightarrow^+ A$, the rewrite relation $B_1 \xrightarrow{+} B_2$ is defined as usual by saying that there exists a position $\mathfrak{p}$ and a substitution σ such that the subformula of B_1 at position $\mathfrak{p}$ is σP and B_2 equals B_1 where the subformula at position $\mathfrak{p}$ is replaced by σA. $\xrightarrow{-}$ is defined similarly. Dowek [15] defined Polarized Sequent Calculus Modulo theory, where the inference rules of the sequent calculus are applied modulo such a polarized rewriting system, as in for instance in $\vdash\!\wedge\ \dfrac{\Gamma \vdash A, \Delta \qquad \Gamma \vdash B \Delta}{\Gamma \vdash C, \Delta}\ C \xrightarrow{+}{}^{*} A \wedge B$

With Kirchner [10], we proved the equivalence of Polarized Sequent Calculus Modulo Theory to a sequent calculus where polarized rewriting rules are applied only on literals, using explicit rules. This calculus, Polarized Unfolding Sequent Calculus, is almost the calculus PUSC$^\perp$ presented in Figure 3. The only difference is that all formulas are put on the left of the sequent in PUSC$^\perp$. We denote by $\Gamma \vdash_{\mathcal{R}}$ the fact that $\Gamma \vdash$ can be proved in PUSC$^\perp$ using the polarized rewriting system $\mathcal{R}$.

We can translated polarized rewriting rules as formulas with selection, and see PUSC$^\perp$ as an instance of LKF$^\perp$. We first consider how to translate formulas of the right-hand side of polarized rewriting rules. We polarize them by choosing negative connectives for $\vee$ and $\wedge$ and, to unchain the introduction of the universal quantifier, we introduce delays. This gives the translation:

$$
\begin{array}{rlrl}
|L| = L & \text{when } L \text{ is } \top,\ \bot \text{ or a literal} & & \\
|A \wedge B| = |A| \wedge^- |B| & & |A \vee B| = |A| \vee^- |B| & \\
|\exists x.\ A| = \exists x.\ |A| & & |\forall x.\ A| = \forall x.\ \delta^- |A| &
\end{array}
$$

$$\widehat{\vdash}\ \frac{}{\Gamma, L, L^{\perp} \vdash}$$

$$\top\vdash\ \frac{\Gamma \vdash}{\Gamma, \top \vdash} \qquad \bot\vdash\ \frac{}{\Gamma, \bot \vdash}$$

$$\vee\vdash\ \frac{\Gamma, A \vdash \qquad \Gamma, B \vdash}{\Gamma, A \vee B \vdash} \qquad \wedge\vdash\ \frac{\Gamma, A, B \vdash}{\Gamma, A \wedge B \vdash}$$

$$\exists\vdash\ \frac{\Gamma, A \vdash}{\Gamma, \exists x.\ A \vdash}\ x \text{ not free in } \Gamma \qquad \forall\vdash\ \frac{\Gamma, \forall x.\ A, \{t/x\}A \vdash}{\Gamma, \forall x.\ A \vdash}$$

$$\uparrow^{-}\vdash\ \frac{\Gamma, P, A \vdash}{\Gamma, P \vdash}\ P \xrightarrow{-} A \qquad \uparrow^{+}\vdash\ \frac{\Gamma, \neg P, A^{\perp} \vdash}{\Gamma, \neg P \vdash}\ P \xrightarrow{+} A$$

Fig. 3. The sequent calculus PUSC$^{\perp}$

Definition 3. – *Given a negative rewriting rule $P \to^{-} A$ where the free variables of P are $x_1, \ldots, x_n$, its translation as a formula with selection is $[\![P \to^{-} A]\!] = \forall x_1.\ \ldots \forall x_n.\ \underline{\neg P} \vee^{+} \delta^{-}|A|$.*

– *Given a positive rewriting rule $P \to^{+} A$ where the free variables of P are $x_1, \ldots, x_n$, its translation as a formula with selection is $[\![P \to^{+} A]\!] = \forall x_1.\ \ldots \forall x_n.\ \underline{P} \vee^{+} \delta^{-}|A^{\perp}|$.*

The translation $[\![\mathcal{R}]\!]$ of a polarized rewriting system $\mathcal{R}$ is the multiset of the translation of its rules.

Definition 4. *Let $P_1, \ldots, P_n$ be a multiset of formulas whose root is $\forall$ or $\bot$ or that are literals, and let $N_1, \ldots, N_m$ be a multiset of non-literal formulas whose root is neither $\forall$ nor $\bot$, then the translation of the PUSC$^{\perp}$ sequent $P_1, \ldots, P_n, N_1, \ldots, N_m \vdash$ modulo the rewriting system $\mathcal{R}$ is the LKF$^{\perp}$ sequent $[\![\mathcal{R}]\!], |P_1|, \ldots, |P_n| \Uparrow |N_1|, \ldots, |N_m| \vdash$.*

Theorem 6. *$P_1, \ldots, P_n, N_1, \ldots, N_m \vdash_{\mathcal{R}}$ iff $[\![\mathcal{R}]\!], |P_1|, \ldots, |P_n| \Uparrow |N_1|, \ldots, |N_m| \vdash$.*

Proof. Proof in both calculi corresponds almost exactly. $\top\vdash$, $\wedge\vdash$, $\vee\vdash$ and $\exists\vdash$ in PUSC$^{\perp}$ corresponds exactly to $\Uparrow\top\vdash$, $\Uparrow\wedge\vdash$, $\Uparrow\vee\vdash$ and $\Uparrow\exists\vdash$ in LKF$^{\perp}$, except that if the root of the subformulas in the premise(s) is $\forall$ or $\bot$, or if they are literals, they have to be put on the left hand side of $\Uparrow$ using $\xleftarrow{\Uparrow}$. The translation of $\bot\vdash$ corresponds to $\Downarrow\bot\vdash$, except that the latter can only be applied when there is no formula with negative polarity:

$$\bot\vdash\ \frac{}{P_1, \ldots, \bot, \ldots, P_n \vdash} \quad \text{becomes} \quad \text{Focus}\ \frac{\Downarrow\bot\vdash\ \dfrac{}{[\![\mathcal{R}]\!], |P_1|, \ldots, \bot, \ldots, |P_n| \Downarrow \bot \vdash}}{[\![\mathcal{R}]\!], |P_1|, \ldots, \bot, \ldots, |P_n| \Uparrow \vdash}\ .$$

Similarly, $\forall\vdash$ corresponds to $\Downarrow\forall\vdash$, with the proviso that there are no formulas with negative polarity:

$$\forall\vdash\ \frac{P_1, \ldots, P_n, \forall x.\ A, \{t/x\}A \vdash}{P_1, \ldots, P_n, \forall x.\ A \vdash} \quad \text{becomes}$$

$$
\dfrac{\dfrac{\dfrac{[\![\mathcal{R}]\!], |P_1|, \dots, |P_n|, \forall x.\ \delta^-|A| \Uparrow |\{t/x\}A| \vdash}{[\![\mathcal{R}]\!], |P_1|, \dots, |P_n|, \forall x.\ \delta^-|A| \Downarrow \delta^-|\{t/x\}A| \vdash}\ \text{Release}}{[\![\mathcal{R}]\!], |P_1|, \dots, |P_n|, \forall x.\ \delta^-|A| \Downarrow \forall x.\ \delta^-|A| \vdash}\ {\Downarrow}\forall{\vdash}}{[\![\mathcal{R}]\!], |P_1|, \dots, |P_n|, \forall x.\ \delta^-|A| \Uparrow \vdash}\ \text{Focus}
$$

with an extra $\overset{\Uparrow}{\leftarrow}$ step if the root of $\{t/x\}A$ is $\forall$ or $\bot$ or if it is a literal.

For the unfolding rules, if P rewrites positively to A, then there exists a rule $Q \rightarrow^+ B$ and a substitution θ such that $P = \theta Q$ and $A = \theta B$. This rule corresponds to a formula $[\![Q \rightarrow^+ B]\!] = \overline{\forall x.}\ \underline{Q} \vee^+ \delta^-|B^\perp|$. Always with the proviso that there is no formula with negative polarity, let $\Gamma = [\![\mathcal{R}']\!], \overline{\forall x.}\ \underline{Q} \vee^+ \delta^-|B^\perp|, |P_1|, \dots, |P_n|, \neg P$, then $\uparrow^+\vdash$ therefore corresponds to

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \Downarrow \underline{P} \vdash}\ \widehat{\Downarrow}{\vdash} \qquad \dfrac{\Gamma \Uparrow |A^\perp| \vdash}{\Gamma \Downarrow \delta^-|A^\perp| \vdash}\ \text{Release}}{\Gamma \Downarrow \underline{\theta Q} \vee^+ \delta^-|\theta B^\perp| \vdash}\ {\Downarrow}{\vee}{\vdash}}{\Gamma \Downarrow \overline{\forall x.}\ \underline{Q} \vee^+ \delta^-|B^\perp| \vdash}\ {\Downarrow}\forall{\vdash}}{\Gamma \Uparrow \vdash}\ \text{Focus}
$$

with an extra $\overset{\Uparrow}{\leftarrow}$ step if the root of $|A^\perp|$ is $\forall$ or $\bot$, or if it is a literal. Reciprocally, given a sequent $[\![\mathcal{R}']\!], \overline{\forall x.}\ \underline{Q} \vee^+ \delta^-|B^\perp|, \Gamma \Uparrow\vdash$, if we apply a Focus on $\overline{\forall x.}\ \underline{Q} \vee^+ \delta^-|B^\perp|$, the derivation is necessarily of the same shape than above, so that there must be a literal $\neg\theta Q$ in Γ. The derivation therefore corresponds to an unfolding of $\neg\theta Q$ into $|\theta B^\perp|$.

The case of a negative rewriting is dual.

There remains to be proved that, in PUSC$^\perp$, the rules $\bot\vdash$,$\forall\vdash$ and $\uparrow\vdash$ can be delayed until the other rules are no longer applicable. This can be done by showing that these rules permute with the other ones. Note that this fact can be related with the strategy used in Tamed [4], a tableaux method based on Deduction Modulo Theory, where rules for universal quantifiers and for rewriting are applied when no other rules can be. $\square$

Let us now consider the subcase where the rewriting rules are clausal, according to the terminology of Dowek [17], e.g. they are of the form $P \rightarrow^- \ulcorner C \urcorner$ or $P \rightarrow^+ \neg(\ulcorner C \urcorner)$ for some clause C. In that case, the resolution method based on Deduction Modulo Theory [18] can be refined into what is called Polarized Resolution Modulo theory [17], whose rules are given in Fig. 4.

Consequently, Theorem 1 shows us that Polarized Resolution Modulo theory is equivalent to PUSC$^\perp$, and therefore to Polarized Sequent Calculus Modulo theory, as shown by Dowek [17]. The Extended Narrowing rules corresponds in fact to Resolution with Selection where the clause with selection corresponds to the translation of a rewriting rule, which means that its left-hand side is selected. For instance, $\text{Ext. Narr.}^- \ \dfrac{P \cup C}{\sigma(D \cup C)}\ \sigma = mgu(P, Q),\ Q \rightarrow^- D$ corresponds to $\text{Resolution with Selection}\ \dfrac{\underline{\neg Q} \cup D \qquad P \cup C}{\sigma(D \cup C)}\ \sigma = mgu(P, Q)$.

$$\textsf{Resolution}\ \frac{P \cup C \qquad \neg Q \cup D}{\sigma(C \cup D)}\ {}^{a} \qquad \textsf{Factoring}\ \frac{L \cup K \cup C}{\sigma(L \cup C)}\ \sigma = mgu(L, K)$$

$$\textsf{Ext. Narr.}^{-}\ \frac{P \cup C}{\sigma(D \cup C)}\ {}^{a},\ Q \rightarrow^{-} D \qquad \textsf{Ext. Narr.}^{+}\ \frac{\neg Q \cup D}{\sigma(C \cup D)}\ {}^{a},\ P \rightarrow^{+} \neg C$$

[a] $\sigma = mgu(P, Q)$

Fig. 4. Inference rules of Polarized Resolution Modulo theory

Deduction Modulo Theory is not always complete. This is the case only if the cut rule is admissible in Polarized Sequent Calculus Modulo theory. It holds for some particular theories, e.g. Simple Type Theory [18] and arithmetics [20]. There are some techniques that ensures this properties [23, 19, 16, 8]. We even proved that any theory can be presented by a rewriting system admitting the cut rule [7]. As presented with Dowek [9], the fact that completeness is not proved once for all, but needs to be proved for each particular theory, is essential. Indeed, if a theory is presented entirely by rewriting rules, completeness implies the consistency of the theory, since no rule can be applied on the empty set of clauses. Consequently, the proof of the completeness cannot be easier than the proof of consistency of the theory, and, according to Gödel, cannot be proven in the theory itself.

We can go a step further and benefit from focusing to decompose the right-hand side formula after an unfolding has occurred. This leads to what Brauner, Houtmann, and Kirchner [5] called Superdeduction. Houtmann [24] studied the links between Superdeduction and focusing, but not with the idea that the rules themselves should be considered as polarized formulas. To link Superdeduction with LKF$^{\perp}$, we just need to change the translation of rewriting rules in order to ensure that the right-hand side is decomposed as much as possible. This is done by suppressing the negative delay δ^{-} and trying to stay in synchronous (i.e. focused) phase by using positive connectives, until we reach a $\exists$ quantifier, after what we try to stay in the asynchronous phase. Note, however, that literals are always given a negative polarization. We introduce the positive translation of a formula:

$$\begin{array}{rlrl}
\lmoustache L \rmoustache & = L & & \text{when } L \text{ is } \top,\ \bot \text{ or a literal} \\
\lmoustache A \wedge B \rmoustache & = \lmoustache A \rmoustache \wedge^{+} \lmoustache B \rmoustache & \lmoustache A \vee B \rmoustache & = \lmoustache A \rmoustache \vee^{+} \lmoustache B \rmoustache \\
\lmoustache \exists x.\ A \rmoustache & = \exists x.\ |A| & \lmoustache \forall x.\ A \rmoustache & = \forall x.\ \lmoustache A \rmoustache
\end{array}$$

and the translation of rewrite rules becomes:

$$\llbracket P \rightarrow^{-} A \rrbracket = \overline{\forall x.}\ \underline{\neg P} \vee^{+} \lmoustache A \rmoustache$$

$$\llbracket P \rightarrow^{+} A \rrbracket = \overline{\forall x.}\ \underline{P} \vee^{+} \lmoustache A^{\perp} \rmoustache$$

The synthetic rules given by the translation of rewriting rules correspond exactly to the superrules of Superdeduction.

4.3 Beyond Deduction Modulo Theory

Example 1. Let us recall the set of clauses of the introduction:

$$\underline{\neg X \in \mathcal{P}(Y)} \cup \underline{\neg Z \in X} \cup Z \in Y \tag{1}$$

$$\underline{X \in \mathcal{P}(Y)} \cup d(X,Y) \in X \tag{2}$$

$$\underline{X \in \mathcal{P}(Y)} \cup \underline{\neg d(X,Y) \in Y} \tag{3}$$

Note that this example is not covered by Ordered Resolution with Selection, at least not if a simplification ordering is used, because we cannot have $X \in \mathcal{P}(Y) \succ \delta(X,Y) \in X$ since with $\theta = \{X \mapsto \mathcal{P}(Z); Y \mapsto Z\}$ their instances are ordered in the wrong direction: $\mathcal{P}(Z) \in \mathcal{P}(Z) \prec \delta(\mathcal{P}(Z), Z) \in \mathcal{P}(Z)$.

The synthetic rules of the example in the introduction corresponds to the derivations when one of the clauses is focused. For instance, if Γ is $\forall X\ Y\ Z.\ \underline{\neg X \in \mathcal{P}(Y)} \vee^{+} \underline{\neg Z \in X} \vee^{+} Z \in Y, \Delta, u \in \mathcal{P}(v), t \in u$, then

$$\text{Focus}\ \dfrac{\Downarrow\forall\vdash\ \dfrac{\Downarrow\vee\vdash\ \dfrac{\widehat{\Downarrow}\vdash\ \dfrac{}{\Gamma \Downarrow \underline{\neg u \in \mathcal{P}(v)} \vdash} \qquad \widehat{\Downarrow}\vdash\ \dfrac{}{\Gamma \Downarrow \underline{\neg t \in u} \vdash} \qquad \text{Release}\ \dfrac{\overset{\Uparrow}{\leftarrow}\ \dfrac{\Gamma, t \in v \Uparrow \vdash}{\Gamma \Uparrow t \in v \vdash}}{\Gamma \Downarrow t \in v \vdash}}{\Gamma \Downarrow \underline{\neg u \in \mathcal{P}(v)} \vee^{+} \underline{\neg t \in u} \vee^{+} t \in v \vdash}}{\Gamma \Downarrow \forall X\ Y\ Z.\ \underline{\neg X \in \mathcal{P}(Y)} \vee^{+} \underline{\neg Z \in X} \vee^{+} Z \in Y \vdash}}{\forall X\ Y\ Z.\ \underline{\neg X \in \mathcal{P}(Y)} \vee^{+} \underline{\neg Z \in X} \vee^{+} Z \in Y, \Delta, u \in \mathcal{P}(v), t \in u \Uparrow \vdash}$$

leads to the synthetic rule

$$(1)\ \dfrac{\Delta, u \in \mathcal{P}(v), t \in u, t \in v \Uparrow \vdash}{\Delta, u \in \mathcal{P}(v), t \in u \Uparrow \vdash}$$

On the resolution side, if we consider the ground instances of Resolution with Selection, we have:

$$\text{R.w.S.}\ \dfrac{\underline{\neg X \in \mathcal{P}(Y)} \cup \underline{\neg Z \in X} \cup Z \in Y \qquad u \in \mathcal{P}(v) \cup C \qquad t \in u \cup D}{t \in v \cup C \cup D}$$

$$\text{R.w.S.}\ \dfrac{\underline{X \in \mathcal{P}(Y)} \cup d(X,Y) \in X \qquad \neg u \in \mathcal{P}(v) \cup C}{d(u,v) \in u \cup C}$$

$$\text{R.w.S.}\ \dfrac{\underline{X \in \mathcal{P}(Y)} \cup \underline{\neg d(X,Y) \in Y} \qquad \neg u \in \mathcal{P}(v) \cup C \qquad d(u,v) \in v \cup D}{C \cup D}$$

hence the derived rules given in the introduction.

The question that remains is how we can prove the completeness of such a selection. We can in fact reduce it to the completeness for several rewriting systems in Deduction Modulo Theory.

Definition 5 (Singleton subselection). *Given a selection function $\mathcal{S}$, the selection function $\mathcal{S}_1$ is a singleton subselection of $\mathcal{S}$ if*

– *$\mathcal{S}_1(C) \subseteq \mathcal{S}(C)$ for all C*
– *if $\mathcal{S}(C) \neq \emptyset$ then $card(\mathcal{S}_1(C)) = 1$*

Example 2.

$$\neg X \in \mathcal{P}(Y) \cup \underline{\neg Z \in X} \cup Z \in Y$$
$$\underline{X \in \mathcal{P}(Y)} \cup d(X,Y) \in X$$
$$\underline{X \in \mathcal{P}(Y)} \cup \neg d(X,Y) \in Y$$

is a singleton subselection of Example 1.

Theorem 7. *Resolution with input selection $\mathcal{S}$ is complete iff for all singleton subselection $\mathcal{S}_1$ of $\mathcal{S}$, Resolution with input selection $\mathcal{S}_1$ is complete*

Proof. See Appendix.

In particular, to prove completeness if we have a clause $\underline{L_1} \cup \ldots \cup \underline{L_n} \cup K_1 \cup \ldots \cup K_m$, one can prove completeness of the n singleton subselection

$$\underline{L_1} \cup \ldots \cup L_n \cup K_1 \cup \ldots \cup K_m$$
$$\vdots$$
$$L_1 \cup \ldots \cup \underline{L_n} \cup K_1 \cup \ldots \cup K_m$$

which correspond, according to Section 4.2, to the n rewriting systems

$$L_1 \rightarrow^{\pm} L_2 \vee \cdots \vee L_n \vee K_1 \vee \cdots \vee K_m$$
$$\vdots$$
$$L_n \rightarrow^{\pm} L_1 \vee \cdots \vee L_{n-1} \vee K_1 \vee \cdots \vee K_m$$

in Deduction Modulo Theory.

Conclusion and Further Work

We generalized focusing and resolution with selection, proved that they corresponds, and showed how known calculi are instances of this framework, namely usual focusing, hyperresolution, Deduction Modulo Theory and Superdeduction. These notable results raise the following new areas of investigations.

First, we need to study how to apply selection also in the generated clauses. This should allow us to cover the cases of Ordered Resolution with Selection and of Semantic Resolution in the first-order case. Dually, in the sequent calculus part, this would correspond to the possibility to dynamically add selection in formulas of subderivations. This could probably be linked with the work of Deplagne and Kirchner [14] where rewrite rules corresponding to induction hypotheses are dynamically added in the rewriting system of a sequent calculus for Deduction Modulo Theory.

Another worthwhile point is how equality should be handled in our framework. In particular, it would be interesting to see how paramodulation calculi, in particular superposition, can be embedded into a sequent calculus.

Bibliography

[1] Andreoli, J.M.: Logic programming with focusing proofs in linear logic. Journal of Logic and Computation 2(3), 297–347 (1992)

[2] Bachmair, L., Ganzinger, H.: Rewrite-based equational theorem proving with selection and simplification. Journal of Logic and Computation 4(3), 1–31 (1994)

[3] Bachmair, L., Ganzinger, H.: Resolution theorem proving. In: Robinson, J.A., Voronkov, A. (eds.) Handbook of Automated Reasoning, pp. 19–99. Elsevier and MIT Press (2001)

[4] Bonichon, R., Hermant, O.: A semantic completeness proof for TaMed. In: Hermann, M., Voronkov, A. (eds.) LPAR. LNCS, vol. 4246, pp. 167–181. Springer (2006)

[5] Brauner, P., Houtmann, C., Kirchner, C.: Principle of superdeduction. In: Ong, L. (ed.) Proceedings of LICS. pp. 41–50 (jul 2007), http://doi.ieeecomputersociety.org/10.1109/LICS.2007.37

[6] Burel, G.: Embedding deduction modulo into a prover. In: Dawar, A., Veith, H. (eds.) CSL. LNCS, vol. 6247, pp. 155–169. Springer (2010)

[7] Burel, G.: From axioms to rewriting rules (2013), available on author's web page

[8] Burel, G.: Cut admissibility by saturation. In: Dowek, G. (ed.) RTA-TLCA 2014. LNCS, vol. 8560, pp. 124–138. Springer (2014)

[9] Burel, G., Dowek, G.: How can we prove that a proof search method is not an instance of another? In: LFMTP'09. pp. 84–87. ACM International Conference Proceeding Series, ACM (2009)

[10] Burel, G., Kirchner, C.: Regaining cut admissibility in deduction modulo using abstract completion. Information and Computation 208(2), 140–164 (2010)

[11] Buss, S.R. (ed.): Handbook of proof theory. Studies in logic and the foundations of mathematics, Elsevier, Amsterdam (1998)

[12] Chaudhuri, K., Pfenning, F.: A focusing inverse method theorem prover for first-order linear logic. In: Nieuwenhuis, R. (ed.) CADE. LNCS, vol. 3632, pp. 69–83. Springer (2005)

[13] Chihani, Z., Libal, T., Reis, G.: The proof certifier checkers. In: TABLEAUX. LNCS, vol. 9323, pp. 201–210. Springer, Wroclaw, Poland (2015), https://hal.inria.fr/hal-01208333

[14] Deplagne, E., Kirchner, C.: Induction as deduction modulo. Rapport de recherche, LORIA (Nov 2004), http://www.loria.fr/publications/2004/A04-R-468/A04-R-468.ps

[15] Dowek, G.: What is a theory? In: Alt, H., Ferreira, A. (eds.) STACS. LNCS, vol. 2285, pp. 50–64. Springer (2002)

[16] Dowek, G.: Truth values algebras and proof normalization. In: Altenkirch, T., McBride, C. (eds.) TYPES. LNCS, vol. 4502, pp. 110–124. Springer (2006)

[17] Dowek, G.: Polarized resolution modulo. In: Calude, C.S., Sassone, V. (eds.) IFIP TCS. IFIP AICT, vol. 323, pp. 182–196. Springer (2010)
[18] Dowek, G., Hardin, T., Kirchner, C.: Theorem proving modulo. Journal of Automated Reasoning 31(1), 33–72 (2003)
[19] Dowek, G., Werner, B.: Proof normalization modulo. The Journal of Symbolic Logic 68(4), 1289–1316 (2003)
[20] Dowek, G., Werner, B.: Arithmetic as a theory modulo. In: Giesl, J. (ed.) RTA. LNCS, vol. 3467, pp. 423–437. Springer (2005)
[21] Farooque, M., Graham-Lengrand, S., Mahboubi, A.: A bisimulation between DPLL(T) and a proof-search strategy for the focused sequent calculus. In: Momigliano, A., Pientka, B., Pollack, R. (eds.) LFMTP. pp. 3–14. ACM (2013), http://doi.acm.org/10.1145/2503887
[22] Goubault-Larrecq, J.: A note on the completeness of certain refinements of resolution. Research Report LSV-02-8, Laboratoire Spécification et Vérification, ENS Cachan, France (2002)
[23] Hermant, O.: Méthodes Sémantiques en Déduction Modulo. Ph.D. thesis, École Polytechnique (2005)
[24] Houtmann, C.: Axiom directed focusing. In: Berardi, S., Damiani, F., de'Liguoro, U. (eds.) Types for Proofs and Programs. LNCS, vol. 5497, pp. 169–185. Springer (2008)
[25] Liang, C., Miller, D.: Focusing and polarization in linear, intuitionistic, and classical logics. Theoretical Computer Science 410(46), 4747–4768 (2009), abstract Interpretation and Logic Programming: In honor of professor Giorgio Levi
[26] McLaughlin, S., Pfenning, F.: Imogen: Focusing the polarized inverse method for intuitionistic propositional logic. In: Cervesato, I., Veith, H., Voronkov, A. (eds.) LPAR. Lecture Notes in Computer Science, vol. 5330, pp. 174–181. Springer (2008)
[27] Miller, D., Volpe, M.: Focused labeled proof systems for modal logic. In: Davis, M., Fehnker, A., McIver, A., Voronkov, A. (eds.) LPAR. LNCS, vol. 9450, pp. 266–280. Springer (2015), http://dx.doi.org/10.1007/978-3-662-48899-7
[28] Robinson, J.A.: Automatic deduction with hyper-resolution. International Journal of Computer Mathematics 1, 227–234 (1965)
[29] Slagle, J.R.: Automatic theorem proving with renamable and semantic resolution. J. ACM 14(4), 687–697 (1967)
[30] Wos, L., Robinson, G.A., Carson, D.F.: Efficiency and completeness of the set of support strategy in theorem proving. J. ACM 12(4), 536–541 (1965)

A Proofs of the lemmas

Proof (of Lemma 1). By induction on the derivation $\Gamma, C_1, \ldots, C_n \overset{\succ}{\rightsquigarrow}^* \emptyset$, generalizing on Γ.

If $\emptyset \in \Gamma$ then trivially $\Gamma, C_1 \cup D, \ldots, C_n \cup D \rightsquigarrow^* \emptyset$.

Otherwise there exists C_{n+1} such that $\Gamma, C_1, \ldots, C_n \rightsquigarrow C_{n+1}$ and $\Gamma, C_1, \ldots, C_n, C_{n+1} \rightsquigarrow^* \emptyset$. There are two cases:

– C_{n+1} is derived using other clauses than one of the C_i. We therefore have $\Gamma, C_1 \cup D, \ldots, C_n \cup D \rightsquigarrow C_{n+1}$.
  We can apply the induction hypothesis on $\Gamma, C_1, \ldots, C_n, C_{n+1} \rightsquigarrow^* \emptyset$, which can be viewed as $\Gamma, C_{n+1}, C_1, \ldots, C_n \rightsquigarrow^* \emptyset$, since by weakening $\Gamma, C_{n+1}, D \rightsquigarrow^* \emptyset$. We obtain $\Gamma, C_{n+1}, C_1 \cup D, \ldots, C_n \cup D \rightsquigarrow^* \emptyset$. By definition of $\rightsquigarrow^*$ we therefore have $\Gamma, C_1 \cup D, \ldots, C_n \cup D \rightsquigarrow^* \emptyset$.
– At least one of the parents of C_{n+1} is some C_i. Since no literal is selected in the C_i, they can only be side clauses of Resolution with Selection, or any clause in Resolution or Factoring. We can therefore derive $C_{n+1} \cup D$ from $\Gamma, C_1 \cup D, \ldots, C_n \cup D$ with the same inference rule that produced C_n. We can apply the induction hypothesis on $\Gamma, C_1, \ldots, C_n, C_{n+1} \rightsquigarrow^* \emptyset$ which gives us $\Gamma, C_1 \cup D, \ldots, C_n \cup D, C \cup D \rightsquigarrow^* \emptyset$. Hence $\Gamma, C_1 \cup D, \ldots, C_n \cup D \rightsquigarrow^* \emptyset$.

□

Proof (of Lemma 2). By induction on the derivation $\Gamma, \theta C \rightsquigarrow^* \emptyset$. As in the previous proof, the only interesting case is when the clause θC is used in the first step $\Gamma, \theta C \rightsquigarrow D$ of the derivation. If the first step $\Gamma, \theta C \rightsquigarrow D$ is Factoring, then $D = \sigma\theta C$ where σ is the most general unifier of two literals in θC. We can apply the induction hypothesis using $\sigma\theta$ instead of σ.

Otherwise, if the first step is Resolution with Selection or Resolution; this step produces therefore a clause $D = \sigma(\theta C' \cup D')$ where σ is the most general unifier of a unification problem $L =^? L', \mathcal{P}$ where L is the literal of C used in this step. Let θC be $L \cup \theta C'$ and let C be $L_1 \cup \cdots \cup L_n \cup C'$ where $\theta L_i = L$ for all i. Let $\omega = mgu(L_1, \ldots, L_n)$, then $\theta = \theta'\omega$. First, from C one can derive $\omega L_1 \cup \omega C'$ by repetitively applying Factoring to C. We have $\sigma L = \sigma L'$, hence $\sigma\theta'\omega L_1 = \sigma L'$. Since we rename variables in the premises of the resolution rules, we can assume that θ' does not affect the variables of L'. Consequently, $\sigma\theta'\omega L_1 = \sigma\theta' L'$. The unification problem $\omega L_1 =^? L', \mathcal{P}$ therefore has a most general solution μ, and there exists κ such that $\sigma\theta' = \kappa\mu$.

From $\omega L_1 \cup \omega C'$ and the same other clauses as in the step $\Gamma, \theta C \rightsquigarrow D$, we can therefore derive $\mu(\omega C' \cup D')$. Once again, the variables of D', coming from the other clauses, can be assumed to be distinct of those of C, therefore $D' = \theta' D'$. We have $\sigma(\theta C' \cup D') = \sigma(\theta'\omega C' \cup D') = \sigma\theta'(\omega C' \cup D') = \kappa\mu(\omega C' \cup D)$. We can therefore apply the induction hypothesis, using κ on $\mu(\omega C' \cup D)$ instead of θ on C. □

Proof (of Lemma 3). By induction on the proof $\Gamma, \ulcorner C \urcorner, \ulcorner \sigma C \urcorner \Uparrow\vdash$. If the proof does not begin by focusing on $\ulcorner \sigma C \urcorner$, this is a simple application of the induction hypothesis. Otherwise, let σC be $\underline{K_1} \cup \cdots \cup \underline{K_n} \cup L_1 \cup \cdots \cup L_m$. The proof begins with

$$\mathsf{Focus}\;\dfrac{\Downarrow\forall\vdash\;\dfrac{\Downarrow\vee\vdash\;\dfrac{\cdots \quad \widehat{\Downarrow\vdash}\;\dfrac{}{\Gamma' \Downarrow \theta\underline{K_j} \vdash} \quad \cdots \quad \mathsf{Release}\;\dfrac{\overset{\Uparrow}{\longleftarrow}\;\dfrac{\Gamma', \theta L_k \Uparrow \vdash}{\Gamma' \Uparrow \theta L_k \vdash}}{\Gamma' \Downarrow \theta L_k \vdash} \quad \cdots}{\Gamma' \Downarrow \theta(\underline{K_1} \vee^+ \cdots \vee^+ \underline{K_n} \vee^+ L_1 \vee^+ \cdots \vee^+ L_m) \vdash}}{\Gamma' \Downarrow \overline{\forall x}.\underline{K_1} \vee^+ \cdots \vee^+ \underline{K_n} \vee^+ L_1 \vee^+ \cdots \vee^+ L_m \vdash}}{\Gamma' \Uparrow \vdash} \quad \text{where } \Gamma' = \Gamma, \ulcorner C \urcorner, \ulcorner \sigma C \urcorner$$

and for all $1 \leq j \leq n$ the literal $\theta {K_j}^\perp$ is in Γ.

We know that C is $K_1^1 \cup \cdots K_1^{k_1} \cup \cdots K_n^1 \cup \cdots K_n^{k_n} \cup L_1^1 \cup \cdots L_1^{l_1} \cup \cdots L_m^1 \cup \cdots L_m^{l_m}$ where $\sigma K_i^j = K_i$ and $\sigma L_i^j = L_i$ for all i, j.

We apply the induction hypothesis on $\Gamma', \theta L_k \Uparrow\vdash$, hence we have proofs $\Gamma, \ulcorner C \urcorner, \theta L_k \Uparrow\vdash$ for all k.

We can therefore build the proof

$$\mathsf{Focus}\;\dfrac{\Downarrow\forall\vdash\;\dfrac{\Downarrow\vee\vdash\;\dfrac{\cdots \quad \mathsf{Release}\;\dfrac{\overset{\Uparrow}{\longleftarrow}\;\dfrac{\widehat{\vdash}\;\dfrac{}{\Gamma, \ulcorner C \urcorner, \theta K_i \Uparrow \vdash}}{\Gamma, \ulcorner C \urcorner \Uparrow \theta K_i \vdash}}{\Gamma, \ulcorner C \urcorner \Downarrow \theta\sigma K_i^j \vdash} \quad \cdots \quad \mathsf{Release}\;\dfrac{\overset{\Uparrow}{\longleftarrow}\;\dfrac{\Gamma, \ulcorner C \urcorner, \theta L_i \Uparrow \vdash}{\Gamma, \ulcorner C \urcorner \Uparrow \theta L_i \vdash}}{\Gamma, \ulcorner C \urcorner \Downarrow \theta\sigma L_i^j \vdash} \quad \cdots}{\Gamma, \ulcorner C \urcorner \Downarrow \cdots \vee^+ \theta\sigma K_i^j \vee^+ \cdots \vee^+ \theta\sigma L_i^j \vee^+ \cdots \vdash}}{\Gamma, \ulcorner C \urcorner \Downarrow \overline{\forall x}. \cdots \vee^+ K_i^j \vee^+ \cdots \vee^+ L_i^j \vee^+ \cdots \vdash}}{\Gamma, \ulcorner C \urcorner \Uparrow \vdash}$$

□

Proof (of Lemma 4). By induction on the proof $\Gamma, \ulcorner L \cup C \urcorner, \ulcorner L'^\perp \cup D \urcorner, \ulcorner \sigma(C \cup D) \urcorner \Uparrow\vdash$. If the proof does not begin by focusing on $\ulcorner \sigma(C \cup D) \urcorner$, this is a simple application of the induction hypothesis. Otherwise, let $\sigma(C \cup D)$ be $\underline{K_1} \cup \cdots \cup \underline{K_n} \cup L_1 \cup \cdots \cup L_m$.

The proof begins with

$$\mathsf{Focus}\;\dfrac{\Downarrow\forall\vdash\;\dfrac{\Downarrow\vee\vdash\;\dfrac{\cdots \quad \widehat{\Downarrow\vdash}\;\dfrac{}{\Gamma' \Downarrow \theta\underline{K_j} \vdash} \quad \cdots \quad \mathsf{Release}\;\dfrac{\overset{\Uparrow}{\longleftarrow}\;\dfrac{\Gamma', \theta L_k \Uparrow \vdash}{\Gamma' \Uparrow \theta L_k \vdash}}{\Gamma' \Downarrow \theta L_k \vdash} \quad \cdots}{\Gamma' \Downarrow \theta(\underline{K_1} \vee^+ \cdots \vee^+ \underline{K_n} \vee^+ L_1 \vee^+ \cdots \vee^+ L_m) \vdash}}{\Gamma' \Downarrow \overline{\forall x}.\underline{K_1} \vee^+ \cdots \vee^+ \underline{K_n} \vee^+ L_1 \vee^+ \cdots \vee^+ L_m \vdash}}{\Gamma' \Uparrow \vdash}$$

where $\Gamma' = \Gamma, \ulcorner L \cup C \urcorner, \ulcorner L'^\perp \cup D \urcorner, \ulcorner \sigma(C \cup D) \urcorner$ and for all $1 \leq j \leq n$ the literal $\theta {K_j}^\perp$ is in Γ.

We know that C is $\cdots \cup K_i^1 \cup \cdots K_i^{k_i} \cup \cdots \cup L_j^1 \cup \cdots L_j^{l_j} \cup \cdots$ where i ranges over a subset of $\{1, \ldots, n\}$ and j over a subset of $\{1, \ldots, m\}$, and $\sigma K_i^x = K_i$ and $\sigma L_j^y = L_j$ for all x, y. Likewise, D is $\cdots \cup {K'}_i^1 \cup \cdots {K'}_i^{k_i} \cup \cdots \cup {L'}_j^1 \cup \cdots {L'}_j^{l_j} \cup \cdots$ where i ranges over a subset of $\{1, \ldots, n\}$ and j over a subset of $\{1, \ldots, m\}$, and $\sigma {K'}_i^x = K_i$ and $\sigma {L'}_j^y = L_j$ for all x, y.

We apply the induction hypothesis on $\Gamma', \theta L_k \Uparrow\vdash$, hence we have proof $\Gamma, \ulcorner L \cup C \urcorner, \ulcorner L'^\perp \cup D \urcorner, \theta L_k \Uparrow\vdash$.

Let $\Gamma'' = \Gamma, \ulcorner L \cup C \urcorner, \ulcorner L'^{\perp} \cup D \urcorner$. We can build the following proof of $\Gamma'' \vdash$: first, we focus on $\ulcorner L \cup C \urcorner$.

$$\text{Focus}\;\dfrac{\Downarrow\forall\vdash\;\dfrac{\Downarrow\vee\vdash\;\dfrac{\text{Release}\;\dfrac{\overset{\Uparrow}{\leftarrow}\;\dfrac{\Gamma'', \theta\sigma L \Uparrow \vdash}{\Gamma'' \Uparrow \theta\sigma L \vdash}}{\Gamma'' \Downarrow \theta\sigma L \vdash} \quad \cdots \quad \text{Release}\;\dfrac{\overset{\Uparrow}{\leftarrow}\;\dfrac{\overset{\frown}{\vdash}\;\dfrac{}{\Gamma'', \theta K_i \Uparrow \vdash}}{\Gamma'' \Uparrow \theta K_i \vdash}}{\Gamma'' \Downarrow \theta\sigma K_i^x \vdash} \quad \cdots \quad \text{Release}\;\dfrac{\overset{\Uparrow}{\leftarrow}\;\dfrac{\Gamma'', \theta L_j \Uparrow \vdash}{\Gamma'' \Uparrow \theta L_j \vdash}}{\Gamma'' \Downarrow \theta\sigma L_j^y \vdash} \quad \cdots}{\Gamma'' \Downarrow \theta\sigma(L \cdots \vee^+ K_i^1 \vee^+ \cdots K_i^{k_i} \vee^+ \cdots \vee^+ L_j^1 \vee^+ \cdots L_j^{l_j} \vee^+ \cdots) \vdash}}{\Gamma'' \Downarrow \overline{\forall x}.\ L \cdots \vee^+ K_i^1 \vee^+ \cdots K_i^{k_i} \vee^+ \cdots \vee^+ L_j^1 \vee^+ \cdots L_j^{l_j} \vee^+ \cdots \vdash}}{\Gamma'' \Uparrow \vdash}$$

The right branches are closed by induction hypothesis. On the left branch, we focus on $\ulcorner L'^{\perp} \cup D \urcorner$. Let Γ''' be $\Gamma', \theta\sigma L$.

$$\text{Focus}\;\dfrac{\Downarrow\forall\vdash\;\dfrac{\Downarrow\vee\vdash\;\dfrac{\text{Release}\;\dfrac{\overset{\Uparrow}{\leftarrow}\;\dfrac{\overset{\frown}{\vdash}\;\dfrac{}{\Gamma''', \theta\sigma L'^{\perp} \Uparrow \vdash}}{\Gamma''' \Uparrow \theta\sigma L'^{\perp} \vdash}}{\Gamma''' \Downarrow \theta\sigma L'^{\perp} \vdash} \quad \cdots \quad \text{Release}\;\dfrac{\overset{\Uparrow}{\leftarrow}\;\dfrac{\overset{\frown}{\vdash}\;\dfrac{}{\Gamma''', \theta K_i \Uparrow \vdash}}{\Gamma''' \Uparrow \theta K_i \vdash}}{\Gamma''' \Downarrow \theta\sigma K_i'^{x} \vdash} \quad \cdots \quad \text{Release}\;\dfrac{\overset{\Uparrow}{\leftarrow}\;\dfrac{\Gamma''', \theta L_j \Uparrow \vdash}{\Gamma''' \Uparrow \theta L_j \vdash}}{\Gamma''' \Downarrow \theta\sigma L_j'^{y} \vdash} \quad \cdots}{\Gamma''' \Downarrow \theta\sigma(L'^{\perp} \cdots \vee^+ K_i'^1 \vee^+ \cdots K_i'^{k_i} \vee^+ \cdots \vee^+ L_j'^1 \vee^+ \cdots L_j'^{l_j} \vee^+ \cdots) \vdash}}{\Gamma''' \Downarrow \overline{\forall x}.\ L'^{\perp} \cdots \vee^+ K_i'^1 \vee^+ \cdots K_i'^{k_i} \vee^+ \cdots \vee^+ L_j'^1 \vee^+ \cdots L_j'^{l_j} \vee^+ \cdots \vdash}}{\Gamma''' \Uparrow \vdash}$$

The left branch can be closed because $\theta\sigma L = \theta\sigma L'$. The right branches are closed by induction hypothesis. $\square$

Proof (of Lemma 5). By induction on the proof $\Gamma, \ulcorner \underline{K_1} \cup \ldots \cup \underline{K_n} \cup C \urcorner, \ldots \ulcorner K_i'^{\perp} \cup D_i \urcorner, \ldots, \ulcorner \sigma(C \cup D_1 \cup \cdots$ We follow the same idea as in the proofs of the two precedent lemmas. If the proof does not begin by focusing on $\ulcorner \sigma(C \cup D_1 \cup \cdots \cup D_n) \urcorner$, this is a simple application of the induction hypothesis. Otherwise, let Γ' be $\Gamma, \ulcorner \underline{K_1} \cup \ldots \cup \underline{K_n} \cup C \urcorner, \ldots \ulcorner K_i'^{\perp} \cup D_i \urcorner, \ldots$ and Γ'' be $\Gamma', \ulcorner \sigma(C \cup D_1 \cup \cdots \cup D_n) \urcorner$. Focusing leads us either to sequents $\Gamma'' \Downarrow \underline{\theta K_j} \vdash$, with $\theta K_j{}^{\perp}$ in Γ, or to sequents $\Gamma'', \theta L_k \Uparrow\vdash$ upon which one can apply the induction hypothesis. Let us remark that for each literal L of C or D_i, $\theta\sigma L$ is either one of θK_j or one of θL_k. Therefore, we know how to close proofs of $\Gamma' \Downarrow \theta\sigma L \vdash$ for each, either by induction hypothesis or using $\theta K_j{}^{\perp}$ in Γ.

To build the proof of $\Gamma' \vdash$, we first focus on $\ulcorner K_1'^{\perp} \cup D_1 \urcorner$, instantiating the variables using the substitution $\theta\sigma$. We know how to close the branches coming from D_1, it remains the branch $\Gamma', \theta\sigma K_1'^{\perp} \Uparrow\vdash$.

We do the same, focusing on $\ulcorner K_2'^{\perp} \cup D_2 \urcorner$ then ... then $\ulcorner K_n'^{\perp} \cup D_n \urcorner$ and the remaining branch is $\Gamma', \theta\sigma K_1'^{\perp}, \ldots \theta\sigma K_n'^{\perp} \Uparrow\vdash$.

We can close the proof by focusing on $\ulcorner \underline{K_1} \cup \ldots \cup \underline{K_n} \cup C \urcorner$. Branches coming from C can be closed as before, and the other branches are closed by $\overset{\frown}{\Downarrow\vdash}\;\dfrac{}{\Gamma', \theta\sigma K_1'^{\perp}, \ldots \theta\sigma K_n'^{\perp} \Downarrow \underline{\theta\sigma K_i} \vdash}$ since $\theta\sigma K_i' = \theta\sigma K_i$ for all i. $\square$

Proof (of Theorem 7). Showing that LKF$^{\perp}$ is complete amounts to proving that the cut rule

$$\underset{\smile}{\vdash}\;\dfrac{\Gamma, A \Uparrow \vdash \qquad \Gamma, A^{\perp} \Uparrow \vdash}{\Gamma \Uparrow \vdash}$$

is admissible.

Using the same techniques as in [10], we can try to eliminate cuts using structural cut elimination à la Pfenning. The only problematic case is when we cut around a literal that is used in focused instance in both branches:

$$
\underset{\smile}{\vdash}\,\dfrac{\mathsf{Focus}\,\dfrac{\begin{matrix}\overset{\frown}{\Downarrow}\vdash\,\dfrac{}{\Gamma', A \Downarrow \underline{A^\perp} \vdash} \\ \vdots \\ \Gamma, A \Downarrow E[\underline{A^\perp}] \vdash\end{matrix}}{\Gamma, A \Uparrow \vdash} \qquad \mathsf{Focus}\,\dfrac{\begin{matrix}\overset{\frown}{\Downarrow}\vdash\,\dfrac{}{\Gamma'', A^\perp \Downarrow \underline{A} \vdash} \\ \vdots \\ \Gamma, A^\perp \Downarrow F[\underline{A}] \vdash\end{matrix}}{\Gamma, A^\perp \Uparrow \vdash}}{\Gamma \Uparrow \vdash}
$$

But, since LKF$^\perp$is complete for all singleton subselection, it is complete if one selects only $A^\perp$ in $E[\underline{A^\perp}]$ and only A in $F[\underline{A}]$. Consequently, we know how to eliminate the cut above. $\square$