Linking Focusing and Resolution with Selection

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Abstract

Focusing and selection are techniques that shrink the proof search space for respectively sequent calculi and resolution. To bring out a link between them, we generalize them both: we introduce a sequent calculus where each *occurrence* of an atom can have a positive or a negative polarity; and a resolution method where each literal, whatever its sign, can be selected in input clauses. We prove the equivalence between cut-free proofs in this sequent calculus and derivations of the empty clause in that resolution method. Such a generalization is not semi-complete in general, which allows us to consider complete instances that correspond to theories of any logical strength. We present three complete instances: first, our framework allows us to show that ordinary focusing corresponds to hyperresolution and semantic resolution; the second instance is deduction modulo theory and the related framework called superdeduction; and a new setting extends deduction modulo theory with rewriting rules having several left-hand sides, therefore restricting even more the proof search space.

Keywords automated deduction, proof theory, sequent calculus, refinements of resolution, deduction modulo theory, polarization

1 Introduction

In addition to clever implementation techniques and data structures, a key point that explains the success of state-of-the-art automated theorem provers is the use of calculi that dramatically reduce proof search space. In the last decades, one can highlight the independent developments of two families of techniques. First, in the kind of methods based on resolution, proof search space can be shrunk using ordering and selection techniques. The intuition is to restrict the application of the resolution rule to only some literals in a clause. If equality is considered, this leads to the superposition calculus [2] which is the base calculus of the currently most efficient automated provers for first-order classical logic. Second, in sequent calculi, Andreoli [1] introduced a technique called focusing to reduce non-determinism in the application of sequent-calculus rules. It works by first applying all invertible rules (those whose conclusion is logically equivalent with their premises) and second by chaining the application of non-invertible rules. Originally developed for linear logic, focusing has been extended to intuitionistic and classical first-order logic [28]. Focusing is mostly used in fields where sequent calculi, and related inverse and tableaux methods, are the most accurate proving method. For instance, there exists tools for first-order linear logic [13], for intuitionistic logic [29] and for modal logic [30]. Focusing is also the key ingredient in Miller's ProofCert project aiming at building a universal framework for proof certification [16].

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Despite their apparent lack of relation, we show in this paper that selection in refinements of the resolution calculus and focusing in sequent calculus are in fact strongly related, so that ordinary focusing in classical first-order logic corresponds actually to hyperresolution, where all negative literals are selected in a clause and are resolved at once. This connection is obtained by relaxing both techniques: concerning resolution, we allow any literal of the input clauses to be selected, whatever its sign; for the focusing part, we allow polarization not only of connectives, but also of all occurrences of literals. The main theorem of this paper, Theorem 4.2, shows that the sets of clauses whose insatisfiability can be proved by the resolution method with arbitrary input selection are exactly the sequents that have a cut-free proof in the generalized focusing setting.

This generalization allows us to cover of wider spectrum of proof systems. In particular, this permits to consider systems that search for proofs modulo some theory. Indeed, in real world applications, proof obligations are often verified within one or several theories. This explains the interest in and the success of Satisfiability Modulo Theory tools in the recent years. Embedding a theory in our framework amounts to giving an axiomatic presentation of it where some literals are selected.

Relaxing the conditions for selecting literals makes that our framework is not refutationaly complete in every case. However, this should not be considered as a drawback, but as an essential point to be able to represent efficiently all kind of theories. Indeed, let us consider a proof search method $\mathbb{P}(\mathcal{T})$ parameterized by a theory $\mathcal{T}$. Ideally, $\mathbb{P}(\mathcal{T})$ should be as efficient as a proof search method without theory if it is fed with a formula that is not related to the theory $\mathcal{T}$. In particular, if it tries to refute the true formula $\top$, it should terminate, and with the answer "NO". Let us say that $\mathbb{P}(\mathcal{T})$ is relatively consistent if it is the case. As pointed out with Dowek [10], we cannot have a generic proof that would work for all $\mathcal{T}$ of the completeness of a relatively consistent method $\mathbb{P}(\mathcal{T})$. Indeed, such a proof would imply the consistency of the theory $\mathcal{T}$, and according to Gödel cannot be performed in $\mathcal{T}$ itself. So either the completeness of the proof system is proved once and for all, but it cannot represent theories that are logically at least as strong as the proof of completeness; or it is not complete in general but it can be proved to be complete for particular theories of any logical strength. What is interesting therefore is to give proofs of completeness of $\mathbb{P}(\mathcal{T})$ for particular theories $\mathcal{T}$.

Consequently, we give three instances of our framework, where we can have proofs of completeness. First, as stated above, we link ordinary focusing with hyperresolution, and, in the ground case, with semantic resolution. Second, we show that Deduction Modulo Theory [21] is also a particular instance of this framework, knowing that there exists numerous proof techniques to prove the completeness of Deduction Modulo a particular theory, for instance [9, 19, 22, 26]. Third, we show how completeness in our framework can be reduced to completeness of several instances of Deduction Modulo Theory. To give an intuition about this last part, and to illustrate how much the proof search space can be constrained without losing completeness, let us consider by example the theory defining the powerset:

$$\forall X,\ \forall Y,\ (X \in \mathcal{P}(Y)) \Leftrightarrow (\forall Z,\ (Z \in X) \Rightarrow (Z \in Y))$$

This theory can be put in clausal normal form, using d as a Skolem symbol, and we select (by underlining them) some literals in these clauses[1]:

$$\underline{\neg X \in \mathcal{P}(Y)} \curlyvee \underline{\neg Z \in X} \curlyvee Z \in Y \qquad (1)$$

$$\underline{X \in \mathcal{P}(Y)} \curlyvee d(X,Y) \in X \qquad (2)$$

$$\underline{X \in \mathcal{P}(Y)} \curlyvee \underline{\neg d(X,Y) \in Y} \qquad (3)$$

Using focusing in general, and in our framework in particular, the decomposition of connectives is so restricted that, given an axiom, a proof derivation using this axiom would necessarily have certain shapes. Thus, the axiom can be replaced by new inference rules, called synthetic rules, that are used instead of the derivation of those shapes. See end of Section 2, page 4, for more details. In our framework, this would lead to the following synthetic rules, that can be used in place of the axioms (the explanation how these rules are obtained is given in Section 5.3):

$$(1)\vdash \ \frac{\Delta, u \in \mathcal{P}(v), t \in u, t \in v \vdash}{\Delta, u \in \mathcal{P}(v), t \in u \vdash}$$

$$(2)\vdash \ \frac{}{\Delta, \neg u \in \mathcal{P}(v), d(u,v) \in v \vdash}$$

$$(3)\vdash \ \frac{\Delta, \neg u \in \mathcal{P}(v), d(u,v) \in u \vdash}{\Delta, \neg u \in \mathcal{P}(v) \vdash}$$

The only proof of transitivity of the membership in the powerset is then

$$\exists\vdash\ \dfrac{\wedge\vdash\ \dfrac{(3)\vdash\ \dfrac{(1)\vdash\ \dfrac{(1)\vdash\ \dfrac{(2)\vdash\ \dfrac{}{a \in \mathcal{P}(b), b \in \mathcal{P}(c), \underline{\neg a \in \mathcal{P}(c)}, d(a,c) \in a, d(a,c) \in b, \underline{d(a,c) \in c} \vdash}}{a \in \mathcal{P}(b), \underline{b \in \mathcal{P}(c)}, \neg a \in \mathcal{P}(c), d(a,c) \in a, \underline{d(a,c) \in b} \vdash}}{\underline{a \in \mathcal{P}(b)}, b \in \mathcal{P}(c), \neg a \in \mathcal{P}(c), \underline{d(a,c) \in a} \vdash}}{a \in \mathcal{P}(b), b \in \mathcal{P}(c), \underline{\neg a \in \mathcal{P}(c)} \vdash}}{\underline{a \in \mathcal{P}(b) \wedge b \in \mathcal{P}(c) \wedge \neg a \in \mathcal{P}(c)} \vdash}}{\underline{\exists A.\ \exists B.\ \exists C.\ A \in \mathcal{P}(B) \wedge B \in \mathcal{P}(C) \wedge \neg A \in \mathcal{P}(C)} \vdash}$$

where the active formulas in a sequent are underwaved, and double lines indicates potentially several applications of an inference rule.

On the resolution side, clauses (1) to (3) leads to the following ground derived rules (see also Section 5.3):

$$(1)\ \frac{u \in \mathcal{P}(v) \curlyvee C \qquad t \in u \curlyvee D}{t \in v \curlyvee C \curlyvee D} \qquad (2)\ \frac{\neg u \in \mathcal{P}(v) \curlyvee C}{d(u,v) \in u \curlyvee C}$$

$$(3)\ \frac{\neg u \in \mathcal{P}(v) \curlyvee C \qquad d(u,v) \in v \curlyvee D}{C \curlyvee D}$$

Once again, there is only one proof of transitivity, i.e. starting from the set of clauses $\{a \in \mathcal{P}(b); b \in \mathcal{P}(c); \neg a \in \mathcal{P}(c)\}$:

$$(3)\ \dfrac{\neg a \in \mathcal{P}(c) \qquad (1)\ \dfrac{b \in \mathcal{P}(c) \qquad (1)\ \dfrac{a \in \mathcal{P}(b) \qquad (2)\ \dfrac{\neg a \in \mathcal{P}(c)}{d(a,c) \in a}}{d(a,c) \in b}}{d(a,c) \in c}}{\Box}$$

and we cannot even infer other clauses than those. We let the reader compare with what happens if we used clauses (1) to (3) in resolution, even using the ordered resolution with selection refinement.

In next section, we present the sequent calculus $\mathrm{LKF}^{\perp}$, which extends the calculus LKF of Liang and Miller [28]. Section 3 introduces the resolution method with arbitrary input selection. Then, the main theorem of Section 4, and of this paper, namely Theorem 4.2, establishes the link between the two proof systems. In Section 5 are highlighted particular instances where proof of completeness can be obtained, which also permits to show an *a priori* unsuspected relation between existing proof systems such as for instance hyperresolution and ordinary focusing.

Related work.

As we do here, Goubault-Larrecq [25] proves completeness of several refinement of resolution, including hyperresolution and semantic resolution, using syntactic transformations instead of relying on the construction of a model.

Chaudhuri et al. [14] show that hyperresolution for Horn clauses can be explained as an instance of a sequent calculus for intuitionistic linear logic with focusing where atoms are given a negative polarity.

Farooque et al. [24] developed a sequent calculus, based on focusing, that is able to simulate DPLL($\mathcal{T}$), the most common calculus used in SMT provers. The main difference with our framework is that in [24], the theory is considered as a black box which is called as an oracle. Here, the theory is considered as a first-class citizen.

Within the ProofCert project, resolution proofs can be checked by a kernel built upon a sequent calculus with focusing [16]. Based on this, the tool `Checkers` of Chihani et al. [15] is able to verify proofs coming from automated theorem provers based on resolution such as E-prover. On the contrary of Theorem 4.11, they translate resolution derivations using cuts to get smaller proofs.

Notations and conventions.

We use standard definitions for terms, predicates, propositions (with connectives $\bot, \top, \neg, \wedge, \vee$ and quantifiers $\forall, \exists$),

[1]We use the associative-commutative-idempotent symbol $\curlyvee$ in clauses to distinguish it from the symbol $\vee$ that is used in formulas.

sequents and substitutions. A literal is an atom or its negation. A clause is a set of literals. We will identify a literal with the unit clause containing it. Unless stated otherwise, letters $P, Q, R, P', P_1, \ldots$ denote atoms, $L, K, L', L_1, \ldots$ denote literals, $A, B, A', A_1, \ldots$ denote formulas, $C, D, C', C_1, \ldots$ denote clauses, Γ, Δ denote set of clauses or set of formulas (depending on the context). $A^\perp$ denotes the negation normal form of $\neg A$.

2 Focusing with Polarized Occurrences of Atoms

Focusing was introduced by Andreoli [1] to restrict the non-determinism in a sequent calculus for linear logic. It relies on the alternation of two phases: During the asynchronous phase (sequents with $\Uparrow$), all invertible rules are applied on the formulas of the sequent. Recall that a rule is said invertible if its conclusion implies the conjunction of its premises. During the synchronous phase (sequents with $\Downarrow$), a particular formula is selected —the focus is on it— and all possible non-invertible rules are successively applied on it. This idea has been extended by Liang and Miller [28] to intuitionistic and classical first-order logic. In these, connectives may have invertible and non-invertible versions of their sequent calculus rules. Therefore, one considers in that case two versions of a connective, one called positive when the right introduction rule is non-invertible, and one called negative when it is invertible. Some connectives, i.e. $\exists$ in classical logic, only have a positive version, and dually, others, such as $\forall$ in classical logic, only have a negative version. Given a usual formula, one can decide which version of a connective one wants to use at a particular occurrence, which is called a polarization of the formula.[2] Note that the polarity of a connective does not affect its semantics, it only alters the shape of the sequent calculus proofs. Similarly, one can decide a polarity for each literals. If a literal with negative polarity L is focused on in a branch, then necessarily this branch must be closed, with $L^\perp$ in the same context. (See rule $\widehat{\Downarrow}\vdash$ in Figure 1.) In the ordinary presentation of focusing, this polarity is chosen globally for all occurrences of each atom, and the polarity of $\neg P$ is defined as the inverse of that of P. The polarity of a formula is defined as the polarity of its top connective. Besides, note that to switch the polarity of a formula, e.g. to impose a change of phase, one can prefix it by so-called delays: $\delta^- A$ is negative whatever the polarity of A. Delays can be defined for instance by $\delta^- A = \forall x.\ A$ where x is not free in A, so we do not need them in the syntax and the rules.

Liang and Miller [28] introduce the sequent calculus LKF, and prove it to be complete for classical first-order logic. In Figure 1, we present the calculus $\mathrm{LKF}^\perp$, which is almost the same with the following differences:

- All formulas are put on the left-hand side of the sequent, instead of the right-hand side. Therefore, one does not try to prove a disjunction of formulas, but one tries to refute a conjunction of formulas. This is the same thanks to the dual nature of classical first-order logic, and this helps to be more close to the resolution derivations. Note that, consequently, the focus is on negative formulas, and invertible rules are applied on positive formulas.
- The polarity of atoms is not chosen globally, but each *occurrence* of a literal can have a positive or a negative polarity. In particular, we can have two literals L and $L^\perp$ which are both negative, or both positive. We denote by $\underline{L}$ the fact that the literal L has a negative polarity. To be able to close branches on which we have two positive opposed literals, we add a rule $\widehat{\Uparrow}\vdash$.

[2]Let us note that this notion of polarity is a standard denomination when dealing with focusing, and should not be confused with the more usual but unrelated notion defined by the parity of the negation-depth of a position in a formula.

Asynchronous phase:

$$\widehat{\Uparrow}\vdash\ \frac{}{\Gamma, L, L^\perp \Uparrow \vdash}$$

$$\Uparrow\exists\vdash\ \frac{\Gamma \Uparrow \Delta, A \vdash}{\Gamma \Uparrow \Delta, \exists x.\ A \vdash}\ x \text{ not free in } \Gamma, \Delta$$

$$\Uparrow\vee\vdash\ \frac{\Gamma \Uparrow \Delta, A \vdash \qquad \Gamma \Uparrow \Delta, B \vdash}{\Gamma \Uparrow \Delta, A \vee^+ B \vdash}$$

$$\Uparrow\wedge\vdash\ \frac{\Gamma \Uparrow \Delta, A, B \vdash}{\Gamma \Uparrow \Delta, A \wedge^+ B \vdash}$$

$$\Uparrow\top\vdash\ \frac{\Gamma \Uparrow \Delta \vdash}{\Gamma \Uparrow \Delta, \top \vdash}$$

Synchronous phase:

$$\widehat{\Downarrow}\vdash\ \frac{}{\Gamma, L^\perp \Downarrow \underline{L} \vdash}$$

$$\Downarrow\forall\vdash\ \frac{\Gamma \Downarrow \{t/x\}A \vdash}{\Gamma \Downarrow \forall x.\ A \vdash}$$

$$\Downarrow\vee\vdash\ \frac{\Gamma \Downarrow A \vdash \qquad \Gamma \Downarrow B \vdash}{\Gamma \Downarrow A \vee^- B \vdash}$$

$$\Downarrow\wedge_1\vdash\ \frac{\Gamma \Downarrow A \vdash}{\Gamma \Downarrow A \wedge^- B \vdash} \qquad \Downarrow\wedge_2\vdash\ \frac{\Gamma \Downarrow B \vdash}{\Gamma \Downarrow A \wedge^- B \vdash}$$

$$\Downarrow\bot\vdash\ \frac{}{\Gamma \Downarrow \bot \vdash}$$

$$\mathsf{Store}\ \frac{\Gamma, A \Uparrow \Delta \vdash}{\Gamma \Uparrow A, \Delta \vdash}\ A \text{ negative or literal}$$

$$\mathsf{Focus}\ \frac{\Gamma, A \Downarrow A \vdash}{\Gamma, A \Uparrow \vdash}\ A \text{ negative}$$

$$\mathsf{Release}\ \frac{\Gamma \Uparrow A \vdash}{\Gamma \Downarrow A \vdash}\ A \text{ positive}$$

Figure 1. The sequent calculus $\mathrm{LKF}^\perp$

We denote by $\Gamma \Uparrow \Delta \vdash$ (with Γ or Δ, possibly empty, containing polarized formulas) the fact that there exists a proof of the sequent $\Gamma \Uparrow \Delta \vdash$ in $\mathrm{LKF}^\perp$, that is, a derivation starting from this sequent and whose all branches are closed (by $\widehat{\Downarrow}\vdash$, $\widehat{\Uparrow}\vdash$ or $\Downarrow\bot\vdash$). Thanks to focusing, such a proof has the following shape :

- Since one starts in asynchronous ($\Uparrow$) phase, invertible rules are successively applied to the positive formulas of Δ, until one obtains negative formulas or literals that are put on the left of $\Uparrow$ using Store.
- When no formula appears on the right of $\Uparrow$, then either the branch is closed by $\widehat{\Uparrow}\vdash$; or the focus is put on a negative formula using Focus.

- In the latter case, one is now in synchronous ($\Downarrow$) phase where non-invertible rules are successively applied to the formula upon which the focus is, until either the branch is closed using $\widehat{\Downarrow}\vdash$ or $\Downarrow\bot\vdash$; or one obtains a positive formula and the synchronous phase ends using Release.
- In the latter case, one starts again in the asynchronous phase.

Focusing therefore strongly constraints the shape of possible proofs, and therefore reduces the proof search space. The $\widehat{\Downarrow}\vdash$ in particular imposes to close branches immediately when the focus is on a negative literal, and thus rules out many derivations.

Restricting proof search using focusing leads to what are called synthetic rules (see for instance [14][pp.148–150] where they are called derived rules): given some formula A, a proof putting the focus on A can only have certain shapes, and thus instead of having A in the context, it can be replaced by new rules synthesizing those shapes. For instance, the formula $\forall x.\ \underline{P(x)} \vee^- ((\exists y.\ Q(x,y)) \wedge^- \underline{R(f(x))})$ in a context Γ can only lead to the following derivations when the focus in put on it:

$$
\begin{prooftree}
\AxiomC{}
\LeftLabel{$\widehat{\Downarrow}\vdash$}
\UnaryInfC{$\Gamma \Downarrow \underline{P(t)} \vdash$}
\AxiomC{$\Gamma, Q(t,y) \Uparrow \vdash$}
\LeftLabel{Store}
\UnaryInfC{$\Gamma \Uparrow Q(t,y) \vdash$}
\LeftLabel{$\Uparrow\exists\vdash$}
\UnaryInfC{$\Gamma \Uparrow \exists y.\ Q(t,y) \vdash$}
\LeftLabel{Release}
\UnaryInfC{$\Gamma \Downarrow \exists y.\ Q(t,y) \vdash$}
\LeftLabel{$\Downarrow\wedge_1\vdash$}
\UnaryInfC{$\Gamma \Downarrow (\exists y.\ Q(t,y)) \wedge^- \underline{R(f(t))} \vdash$}
\LeftLabel{$\Downarrow\vee\vdash$}
\BinaryInfC{$\Gamma \Downarrow \underline{P(t)} \vee^- ((\exists y.\ Q(t,y)) \wedge^- \underline{R(f(t))}) \vdash$}
\LeftLabel{$\Downarrow\forall\vdash$}
\UnaryInfC{$\Gamma \Downarrow \forall x.\ \underline{P(x)} \vee^- ((\exists y.\ Q(x,y)) \wedge^- \underline{R(f(x))}) \vdash$}
\LeftLabel{Focus}
\UnaryInfC{$\Gamma \Uparrow \vdash$}
\end{prooftree}
$$

with $P(t)^\perp$ in Γ to be able to apply $\widehat{\Downarrow}\vdash$, and y not free in Γ;

$$
\begin{prooftree}
\AxiomC{}
\LeftLabel{$\widehat{\Downarrow}\vdash$}
\UnaryInfC{$\Gamma \Downarrow \underline{P(t)} \vdash$}
\AxiomC{}
\LeftLabel{$\widehat{\Downarrow}\vdash$}
\UnaryInfC{$\Gamma \Downarrow \underline{R(f(t))} \vdash$}
\LeftLabel{$\Downarrow\wedge_2\vdash$}
\UnaryInfC{$\Gamma \Downarrow (\exists y.\ Q(t,y)) \wedge^- \underline{R(f(t))} \vdash$}
\LeftLabel{$\Downarrow\vee\vdash$}
\BinaryInfC{$\Gamma \Downarrow \underline{P(t)} \vee^- ((\exists y.\ Q(t,y)) \wedge^- \underline{R(f(t))}) \vdash$}
\LeftLabel{$\Downarrow\forall\vdash$}
\UnaryInfC{$\Gamma \Downarrow \forall x.\ \underline{P(x)} \vee^- ((\exists y.\ Q(x,y)) \wedge^- \underline{R(f(x))}) \vdash$}
\LeftLabel{Focus}
\UnaryInfC{$\Gamma \Uparrow \vdash$}
\end{prooftree}
$$

with both $P(t)^\perp$ and $R(f(t))^\perp$ in Γ to be able to apply $\widehat{\Downarrow}\vdash$.

In the first case, Γ is therefore $\forall x.\ \underline{P(x)} \vee^- ((\exists y.\ Q(x,y)) \wedge^- \underline{R(f(x))}), \Delta, P(t)^\perp$; in the second case, it is $\forall x.\ \underline{P(x)} \vee^- ((\exists y.\ Q(x,y)) \wedge^- \underline{R(f(x))}), \Delta, P(t)^\perp, R(f(t))^\perp$. Instead of having the formula in the context, the following two synthetic rules can therefore be used:

$$
\mathsf{Syn1}\ \frac{\Delta, P(t)^\perp, Q(t,y) \Uparrow \vdash}{\Delta, P(t)^\perp \Uparrow \vdash}\ y \text{ not free in } \Delta, P(t)^\perp
$$

$$
\mathsf{Syn2}\ \frac{}{\Delta, P(t)^\perp, R(f(t))^\perp \Uparrow \vdash}
$$

The sequent calculus $\mathrm{LKF}^\perp$ is not complete in general. One of the simplest example of incompleteness is the sequent $\underline{P} \vee^- Q, \underline{\neg P} \vee^- Q, \neg Q \Uparrow\vdash$ which has no proof although $P \vee Q, \neg P \vee Q, \neg Q$ is not satisfiable.

One could choose a global polarization for atoms and use delays to enforce the polarity of occurrences of literals without compromising completeness. However, this would break synchronization phases, and the proof-search space would not be as restricted as in our calculus. For instance, let A be an arbitrary large tautology, and let P be an atom not appearing in A. Obviously, the set of formulas $P \vee A, \neg P$ is satisfiable and thus cannot be refuted. If one consider the polarization where both P and $\neg P$ are negative in the formulas, no inference rule can be applied on the sequent $\underline{P} \vee^- A, \underline{\neg P} \Uparrow\vdash$. However, if one considers a global polarization where for instance P is positive, even if P is enforced to be negative in the first formula using a delay δ^-, then a proof search can begin with

$$
\begin{prooftree}
\AxiomC{}
\LeftLabel{$\widehat{\Downarrow}\vdash$}
\UnaryInfC{$\Gamma, P \Downarrow \underline{\neg P} \vdash$}
\LeftLabel{Focus}
\UnaryInfC{$\Gamma, P \Uparrow \vdash$}
\LeftLabel{Store}
\UnaryInfC{$\Gamma \Uparrow P \vdash$}
\LeftLabel{Release}
\UnaryInfC{$\Gamma \Downarrow \delta^- P \vdash$}
\AxiomC{attempt to refute A}
\UnaryInfC{$\Gamma \Downarrow A \vdash$}
\LeftLabel{$\Downarrow\vee\vdash$}
\BinaryInfC{$\Gamma \Downarrow \delta^- P \vee^- A \vdash$}
\LeftLabel{Focus}
\UnaryInfC{$\delta^- P \vee^- A, \underline{\neg P} \Uparrow \vdash$}
\end{prooftree}
$$

where Γ is $\delta^- P \vee^- A, \underline{\neg P}$. Depending on A, we cannot even warranty that the attempt to refute A will eventually end with a failure. Polarizing occurrences of literals therefore permits to finely tune the proof search space to reduce it while keeping completeness.

3 Resolution with Input Selection

Ordered Resolution with Selection was introduced by Bachmair and Ganzinger [2] (see also Bachmair and Ganzinger [3]) as a complete refinement of resolution. It is parameterized by a Noetherian ordering $\succ$ and a selection function $\mathcal{S}$. If the selection function selects at least a literal, only those can be used in Resolution. Otherwise, only the maximal literals w.r.t. $\succ$ can be used. In the usual setting, the selection function can only select a subset of the *negative* literals of a clause, where by negative literal we mean here the negation of an atomic formula. Here, we relax this condition, but we only use selection in the input clause : in them, any literal can be selected. In the generated clauses however, we impose that $\mathcal{S}(C) = \emptyset$, and we do not consider ordering restrictions. We also allow to have the same input clause several times with different selections. Note that neither Resolution with Input Selection is a generalization of Ordered Resolution with Selection nor the converse. In the following, when speaking about a clause, we talk about a clause equipped with a set of selected literals (which is therefore empty for generated clauses).

The inference rules of Resolution with Input Selection are presented in Fig. 2. Literals that are selected in a clause are underlined. We will see that they indeed correspond to the literals that have a negative polarization in $\mathrm{LKF}^\perp$. As usual, variables are renamed in the clauses to avoid that premises of the inference rules share variables. Note that two clauses with a non-empty selection cannot be resolved together. This is reminiscent of the set-of-support strategy for resolution [34],

$$\text{Resolution with Selection}\ \frac{\underline{K_1} \parr \ldots \parr \underline{K_n} \parr C \qquad {K'_1}^\perp \parr D_1 \qquad \ldots \qquad {K'_n}^\perp \parr D_n}{\sigma(C \parr D_1 \parr \ldots \parr D_n)}$$

- $\mathcal{S}(\underline{K_1} \parr \ldots \parr \underline{K_n} \parr C) = \{K_1; \ldots; K_n\}$
- $\mathcal{S}({K'_i}^\perp \parr D_i) = \emptyset$
- σ is the most general unifier of the simultaneous unification problem $K_1 =^? K'_1, \ldots, K_n =^? K'_n$

$$\text{Resolution}\ \frac{L \parr C \qquad {L'}^\perp \parr D}{\sigma(C \parr D)}$$

- $\mathcal{S}(L \parr C) = \emptyset$
- $\mathcal{S}({L'}^\perp \parr D) = \emptyset$
- σ is the most general unifier of $L =^? L'$

$$\text{Factoring}\ \frac{L \parr L' \parr C}{\sigma(L \parr C)}$$

- $\mathcal{S}(L \parr L' \parr C) = \emptyset$
- σ is the most general unifier of $L =^? L'$

Figure 2. Resolution with Input Selection

by considering that clause with a non-empty selection are outside the set of support.

Definition 3.1 (Resolution derivation)**.** We write $\Gamma \leadsto C$ if C can be derived from some clauses in Γ using the inference rule Resolution with Selection, Resolution, or Factoring presented in Figure 2.

We write $\Gamma \leadsto^* C$ if

- $C \in \Gamma$ or if
- there exists D such that $\Gamma \leadsto D$ and $\Gamma, D \leadsto^* C$.

As usual in resolution methods, the goal is to produce the empty clause $\Box$ starting from a set of clauses Γ to show, since all rules are sound, that Γ is unsatisfiable. Here again, the calculus is not complete in general: from the set of clauses $\underline{P} \parr Q, \underline{\neg P} \parr Q, \neg Q$, no inference rule can be applied: to apply Resolution with Selection, we would need a clause where P, or $\neg P$, is not selected, and Resolution needs two clauses without selection.

4 Focusing is a Conservative Extension of Resolution with Input Selection

To link LKF$^\perp$ with Resolution with Input Selection, we need to indicate how clauses are related to polarized formulas.

Definition 4.1. Given a clause $C = \underline{L_1} \parr \cdots \parr \underline{L_n} \parr K_1 \parr \ldots \parr K_m$ whose free variables are $x_1, \ldots, x_l$ and such that $\mathcal{S}(C) = \{L_1; \ldots; L_n\}$, we define the associated formula $\ulcorner C \urcorner = \forall x_1, \ldots, x_l.\ \underline{L_1} \vee^- \cdots \vee^- \underline{L_n} \vee^- (K_1 \vee^+ \cdots \vee^+ K_m)$. $\ulcorner C \urcorner$ is said to be in clausal form. By extension, $\ulcorner \Gamma \urcorner$ is the set of the formulas associated to the clauses of the set Γ.

The main theorem of this article relates LKF$^\perp$ with Resolution with Input Selection:

Theorem 4.2. *Let Γ be a set of clauses. We have $\ulcorner \Gamma \urcorner \Uparrow\vdash$ iff $\Gamma \leadsto^* \Box$.*

The two direction of the proof are given in the next sections. Due to a lack of space, some proofs are given in the appendix.

4.1 From Focused Proofs to Resolution Derivations

We need a few lemmas to prove the first direction.

Lemma 4.3. *For all set of clauses Γ, for all clauses $C_1, \ldots, C_n$ and D such that $\mathcal{S}(C_i) = \emptyset$ for all i and $\mathcal{S}(D) = \emptyset$, if $\Gamma, C_1, \ldots, C_n \leadsto^* \Box$ and $\Gamma, D \leadsto^* \Box$, then $\Gamma, C_1 \parr D, \ldots, C_n \parr D \leadsto^* \Box$ where $\mathcal{S}(C_i \parr D) = \emptyset$ for all i.*

Corollary 4.4. *For all set of clauses Γ, for all clauses $C_1, \ldots, C_n$ such that $\mathcal{S}(C_i) = \emptyset$ for all i. If $\Gamma, C_i \leadsto^* \Box$ for all i then $\Gamma, C_1 \parr \cdots \parr C_n \leadsto^* \Box$ where $\mathcal{S}(C_1 \parr \cdots \parr C_n) = \emptyset$.*

Lemma 4.5. *For all set of clauses Γ, for all substitutions θ, for all clauses C such that $\mathcal{S}(\theta C) = \emptyset$, if $\Gamma, \theta C \leadsto^* \Box$ then $\Gamma, C \leadsto^* \Box$ where $\mathcal{S}(C) = \emptyset$.*

Theorem 4.6. *If $\ulcorner \Gamma \urcorner \Uparrow\vdash$, then $\Gamma \leadsto^* \Box$.*

Proof. By induction on the proof $\ulcorner \Gamma \urcorner \Uparrow\vdash$. On such a sequent, only two rules can be applied, namely $\widehat{\Uparrow\vdash}$ and Focus. Since $\ulcorner \Gamma \urcorner$ contains only formulas in clausal form, there are only four cases:

- $$\widehat{\vdash}\ \frac{}{\ulcorner \Gamma' \urcorner, L, L^\perp \Uparrow \vdash}$$
  In that case, we can simply apply Resolution on L and $L^\perp$ to derive $\Box$, hence $\Gamma', L, L^\perp \leadsto \Box$.
- The proof focuses on a formula corresponding to the empty clause:
  $$\text{Focus}\ \frac{\Downarrow\bot\vdash\ \dfrac{}{\ulcorner \Gamma \urcorner \Downarrow \bot \vdash}}{\ulcorner \Gamma \urcorner \Uparrow \vdash}$$
  In that case, $\Box$ already belongs to Γ.
- The proof focuses on a formula corresponding to a clause without selection. Because of focusing constraints, the proof is necessarily of the form:
  $$\text{Focus}\ \dfrac{\Downarrow\forall\vdash\ \dfrac{\text{Release}\ \dfrac{\Uparrow\vee\vdash\ \dfrac{\text{Store}\ \dfrac{\ulcorner \Gamma \urcorner, \theta L_1 \Uparrow \vdash}{\ulcorner \Gamma \urcorner \Uparrow \theta L_1 \vdash} \quad \ldots \quad \text{Store}\ \dfrac{\ulcorner \Gamma \urcorner, \theta L_m \Uparrow \vdash}{\ulcorner \Gamma \urcorner \Uparrow \theta L_m \vdash}}{\ulcorner \Gamma \urcorner \Uparrow \theta(L_1 \vee^+ \cdots \vee^+ L_m) \vdash}}{\ulcorner \Gamma \urcorner \Downarrow \theta(L_1 \vee^+ \cdots \vee^+ L_m) \vdash}}{\ulcorner \Gamma \urcorner \Downarrow \overline{\forall x}.L_1 \vee^+ \cdots \vee^+ L_m \vdash}}{\ulcorner \Gamma \urcorner \Uparrow \vdash}$$
  By induction hypothesis, we have derivations of $\Gamma, \theta L_k \leadsto^* \Box$ for all $1 \leq k \leq m$. By Corollary 4.4, we have a derivation $\Gamma, \theta(L_1 \parr \cdots \parr L_k) \leadsto^* \Box$ with nothing selected in $\theta(L_1 \parr \cdots \parr L_k)$. By Lemma 4.5, we have a derivation of $\Gamma \leadsto^* \Box$, with $L_1 \parr \cdots \parr L_k$ in Γ.
- The proof focuses on a formula corresponding to a clause with selection. Because of focusing constraints, the proof is necessarily of the form:
  $$\text{Release}\ \dfrac{\Uparrow\vee\vdash\ \dfrac{\ldots \quad \text{Store}\ \dfrac{\ulcorner \Gamma \urcorner, \theta L_k \Uparrow \vdash}{\ulcorner \Gamma \urcorner \Uparrow \theta L_k \vdash} \quad \ldots}{\ulcorner \Gamma \urcorner \Uparrow \theta(L_1 \vee^+ \cdots \vee^+ L_m) \vdash}}{\ulcorner \Gamma \urcorner \Downarrow \theta(L_1 \vee^+ \cdots \vee^+ L_m) \vdash}$$

$$\text{Focus}\;\cfrac{\Downarrow\forall\vdash\;\cfrac{\Downarrow\vee\vdash\;\cfrac{\cdots \quad \widehat{\Downarrow}\vdash\,\cfrac{}{\ulcorner\Gamma\urcorner \Downarrow \underline{\theta K_j} \vdash} \quad \cdots \quad \vdots}{\ulcorner\Gamma\urcorner \Downarrow \theta(\underline{K_1} \vee^- \cdots \vee^- \underline{K_n} \vee^- (L_1 \vee^+ \cdots \vee^+ L_m) \vdash}}{\ulcorner\Gamma\urcorner \Downarrow \overline{\forall x}.\underline{K_1} \vee^- \cdots \vee^- \underline{K_n} \vee^- (L_1 \vee^+ \cdots \vee^+ L_m) \vdash}}{\ulcorner\Gamma\urcorner \Uparrow \vdash}$$

where $\overline{\forall x}.\underline{K_1} \vee^- \cdots \vee^- \underline{K_n} \vee^- (L_1 \vee^+ \cdots \vee^+ L_m)$ and ${\theta K_j}^\perp$ for all $1 \le j \le n$ are member of $\ulcorner\Gamma\urcorner$.
By induction hypothesis, we have derivations of $\Gamma, \theta L_k \rightsquigarrow^* \Box$ for all $1 \le k \le m$. By Corollary 4.4, we have a derivation $\Gamma, \theta(L_1 \curlyvee \cdots \curlyvee L_k) \rightsquigarrow^* \Box$.
From $\underline{K_1} \curlyvee \cdots \curlyvee \underline{K_n} \curlyvee L_1 \curlyvee \cdots \curlyvee L_m$, since the application of $\widehat{\Downarrow}\vdash$ above imposes that all ${\theta K_j}^\perp$ are in Γ, we can apply Resolution with Selection to obtain $\theta(L_1 \curlyvee \cdots \curlyvee L_m)$, hence a derivation $\Gamma \rightsquigarrow^* \Box$. ◻

4.2 From Resolution Derivations to Focused Proofs

We prove that all inference rules of Resolution with Input Selection are admissible in LKF$^\perp$: if $\Gamma \rightsquigarrow C$ then LKF$^\perp$ proofs of $\ulcorner\Gamma\urcorner, \ulcorner C\urcorner \vdash$ can be turned into proofs of $\ulcorner\Gamma\urcorner \vdash$.

Lemma 4.7. *For all set of formulas Γ, for all clauses C, for all substitution σ, assuming $\sigma\mathcal{S}(C) \subseteq \mathcal{S}(\sigma C)$, if $\Gamma, \ulcorner C\urcorner, \ulcorner \sigma C\urcorner \Uparrow\vdash$, then $\Gamma, \ulcorner C\urcorner \vdash$.*

Corollary 4.8. Factoring *is admissible in LKF$^\perp$.*

Lemma 4.9. Resolution *is admissible in LKF$^\perp$:*
For all set of formulas Γ, for all clauses $L \curlyvee C$ and ${L'}^\perp \curlyvee D$ without selection, if $\sigma = mgu(L, L')$,
if $\Gamma, \ulcorner L \curlyvee C\urcorner, \ulcorner {L'}^\perp \curlyvee D\urcorner, \ulcorner \sigma(C \curlyvee D)\urcorner \Uparrow\vdash$
then $\Gamma, \ulcorner L \curlyvee C\urcorner, \ulcorner {L'}^\perp \curlyvee D\urcorner \Uparrow\vdash$.

Lemma 4.10. Resolution with Selection *is admissible in LKF$^\perp$:*
For all set of formulas Γ, for all clauses $\underline{K_1} \curlyvee \ldots \curlyvee \underline{K_n} \curlyvee C$, ${K'_1}^\perp \curlyvee D_1, \ldots,$ and ${K'_n}^\perp \curlyvee D_n$, where $\mathcal{S}(\underline{K_1} \curlyvee \ldots \curlyvee \underline{K_n} \curlyvee C) = \{K_1; \ldots; K_n\}$, $\mathcal{S}({K'_i}^\perp \curlyvee D_i) = \emptyset$ and σ is the most general unifier of the simultaneous unification problem $K_1 =^? K'_1, \ldots, K_n =^? K'_n$,
if $\Gamma, \ulcorner \underline{K_1} \curlyvee \ldots \curlyvee \underline{K_n} \curlyvee C\urcorner, \ldots \ulcorner {K'_i}^\perp \curlyvee D_i\urcorner, \ldots, \ulcorner \sigma(C \curlyvee D_1 \curlyvee \cdots \curlyvee D_n)\urcorner \Uparrow\vdash$
then $\Gamma, \ulcorner \underline{K_1} \curlyvee \ldots \curlyvee \underline{K_n} \curlyvee C\urcorner, \ldots \ulcorner {K'_i}^\perp \curlyvee D_i\urcorner, \ldots \Uparrow\vdash$.

Theorem 4.11. *For all set of clauses Γ, if $\Gamma \rightsquigarrow^* \Box$, then $\ulcorner\Gamma\urcorner \Uparrow\vdash$.*

Proof. By induction on the length of the derivation $\Gamma \rightsquigarrow^* \Box$. If $\Box$ is in Γ, then we focus on $\ulcorner\Box\urcorner = \bot$ and apply $\Downarrow\bot\vdash$. If the first step is Factoring, we apply Lemma 4.7. If it is Resolution, we apply Lemma 4.9. If it is Resolution with Selection, we apply Lemma 4.10. ◻

Note that the inference rules of Resolution with Input Selection are admissible, but they are not derivable. In particular, the size of the proof in LKF$^\perp$ can be much larger than the resolution derivation, as expected in a cut-free sequent calculus. Using cuts would lead to a more close correspondence between resolution derivations and sequent-calculus proofs, as in [16]. However, we chose to stay in the cut-free fragment to prove that, even in the incomplete case, resolution coincides with cut-free proofs.

5 Complete Instances

5.1 Ordinary Focusing and Semantic Hyperresolution

As said earlier, in LKF, not all occurrences of literals can have an arbitrary polarity. Instead, each atom P is given globally a polarity, and $P^\perp$ has the opposite polarity.

Let us first look at the simple case where atoms are given a positive polarity.

Theorem 5.1 (Corollary of [28, Theorem 17])**.** *If the literals with a positive polarity are exactly the atoms, LKF$^\perp$ is (sound and) complete.*

If we look at the corresponding resolution calculus, Resolution with Selection for this particular instance becomes:

$$\text{R.w.S.}\;\frac{\neg P_1 \curlyvee \ldots \curlyvee \neg P_n \curlyvee C \qquad P'_1 \curlyvee D_1 \quad \ldots \quad P'_n \curlyvee D_n}{\sigma(C \curlyvee D_1 \curlyvee \ldots \curlyvee D_n)}$$

where C and D_i for all i contain only positive literals, and σ is the most general unifier of $P_1 =^? P'_1, \ldots, P_n =^? P'_n$. Note that the clause $\sigma(C \curlyvee D_1 \curlyvee \ldots \curlyvee D_n)$ contains only positive literals, so no literal would be selected in it even if it was an input clause.

Besides, Resolution cannot be applied, since there exists no clause $\neg P \curlyvee C$ with $\mathcal{S}(\neg P \curlyvee C) = \emptyset$.

This corresponding resolution calculus is therefore exactly hyperresolution of Robinson [31]: premises of an inference contains all only positive literals, except one clause whose all negative literals are resolved at once. Theorem 4.2 therefore links ordinary focusing with hyperresolution. Consequently, Theorem 5.1 implies the completeness of hyperresolution.

Chaudhuri et al. [14][Theorem 16] prove a similar result by establishing a correspondence between hyperresolution derivations and proofs in a focused sequent calculus for intuitionistic linear logic, but only considering Horn clauses. In their setting, choosing a negative polarity for atoms leads to SLD resolution, which is the reasoning mechanism of Prolog.

Let us now look at the general case, where atoms are given an arbitrary polarity. Let us stick to the ground case. We first recall a refinement of resolution called Semantic hyperresolution [33][12, Sect. 1.3.5.3]. Let I be a Herbrand interpretation, i.e. a model whose domain is the set of terms interpreted as themselves. Note that I is not assumed to be a model of the input set of clauses (which is fortunate, since one is trying to show that it is unsatisfiable). Given a clause C, the idea of semantic hyperresolution is to resolve all literals of C that are valid in I at once, with clauses whose all literals are not valid in I. This gives the rule:

$$\text{SHR}\;\frac{K_1 \curlyvee \ldots \curlyvee K_n \curlyvee C \qquad {K_1}^\perp \curlyvee D_1 \quad \ldots \quad {K_n}^\perp \curlyvee D_n}{C \curlyvee D_1 \curlyvee \ldots \curlyvee D_n}$$

where for all i, $I \models K_i$ (and thus $I \not\models K_i^\perp$), $I \not\models C$ and $I \not\models D_i$. Note that $I \not\models C \curlyvee D_1 \curlyvee \ldots \curlyvee D_n$.

Semantic hyperresolution for a Herbrand interpretation I can be seen as an instance of Resolution with Input Selection by using the following polarization of atoms: a literal L has a negative polarity iff $I \models L$. In that case, SHR corresponds exactly to Resolution with Selection, and Resolution cannot be applied since we cannot have clauses $P \curlyvee C$ and $\neg P \curlyvee D$ where both P and $\neg P$ are not valid in I.

This particular instance of polarization is in fact the ordinary version of focusing. Indeed, once a global polarity is assigned to each atom, the set of literal whose polarity is negative defines an Herbrand interpretation, and we saw reciprocally how to design a global polarization from the Herbrand interpretation. Theorem 4.2 therefore links ordinary focusing in the ground case with semantic hyperresolution. We know they are complete according to the following theorem:

Theorem 5.2 (Corollary of [28, Theorem 17])**.** *Given a global polarization of atoms, where the polarity of $P^\perp$ is the opposite of that of P, LKF$^\perp$ is (sound and) complete.*

Completeness of LKF therefore leads to proofs of completeness of semantic hyperresolution in the ground case, and hyperresolution in the first-order case, that do not rely on the construction of a model. To our knowledge, in addition to Chaudhuri et al. [14], only Goubault-Larrecq [25] showed a similar result. We also provided a proof similar to the one in this paper, but restricted to Ordered Resolution, without selection of literals [6].

Note that we cannot extend this link between semantic hyperresolution and focusing to the non-ground case. Indeed, in the non-ground case, $I \models L$ iff I is a model for all ground instances of L. Hence, given a literal L such that $I \not\models L$, we can nonetheless have an instance σL such that $I \models \sigma L$; σL should therefore be selected in the generated clause, but our resolution calculus cannot take this into account.

5.2 Deduction Modulo Theory

Deduction Modulo Theory is a framework introduced by Dowek et al. [21] that consists in applying the inference rules of an existing proof system modulo some congruence over formulas. This congruence represents the theory, and it is in general defined by means of rewriting rules. To be expressive enough, these rules are defined not only at the term level, but also for formulas. To get simpler presentations of theories, we distinguish between rewrite rules that can be applied at positive and at negative positions by giving them a polarity[3], where by negative position we mean under a odd number of $\neg$. We therefore have positive rules $P \rightarrow^+ A$ and negative rules $P \rightarrow^- A$ where P is an atom and A an arbitrary formula whose free variables appears in P. Given a rule $P \rightarrow^+ A$, the rewrite relation $B_1 \xrightarrow{+} B_2$ is defined as usual by saying that there exists a position $\mathfrak{p}$ and a substitution σ such that the subformula of B_1 at position $\mathfrak{p}$ is σP and B_2 equals B_1 where the subformula at position $\mathfrak{p}$ is replaced by σA. $\xrightarrow{-}$ is defined similarly. Dowek [18] defined Polarized Sequent Calculus Modulo theory, where the inference rules of the sequent calculus are applied modulo such a polarized rewriting system, as in for instance in $\vdash\wedge \dfrac{\Gamma \vdash A, \Delta \qquad \Gamma \vdash B, \Delta}{\Gamma \vdash C, \Delta}\ C \xrightarrow{+}{}^* A \wedge B$ ⋅ Note that the implicit semantics of a negative rule $P \rightarrow^- A$ is therefore $\overline{\forall x}.\ (P \Rightarrow A)$, whereas the semantics of $P \rightarrow^+ A$ is $\overline{\forall x}.\ (A \Rightarrow P)$, where $\overline{x}$ are the free variables of P.

[3]This polarity must not be confused with the other notions of polarity mentioned in the paper.

$$\overset{\frown}{\vdash}\ \frac{}{\Gamma, L, L^\perp \vdash}$$

$$\top\vdash\ \frac{\Gamma \vdash}{\Gamma, \top \vdash} \qquad\qquad \bot\vdash\ \frac{}{\Gamma, \bot \vdash}$$

$$\vee\vdash\ \frac{\Gamma, A \vdash \qquad \Gamma, B \vdash}{\Gamma, A \vee B \vdash} \qquad\qquad \wedge\vdash\ \frac{\Gamma, A, B \vdash}{\Gamma, A \wedge B \vdash}$$

$$\exists\vdash\ \frac{\Gamma, A \vdash}{\Gamma, \exists x.\ A \vdash}\ x \text{ not free in } \Gamma \qquad \forall\vdash\ \frac{\Gamma, \forall x.\ A, \{t/x\}A \vdash}{\Gamma, \forall x.\ A \vdash}$$

$$\uparrow^-\vdash\ \frac{\Gamma, P, A \vdash}{\Gamma, P \vdash}\ P \xrightarrow{-} A \qquad \uparrow^+\vdash\ \frac{\Gamma, \neg P, A^\perp \vdash}{\Gamma, \neg P \vdash}\ P \xrightarrow{+} A$$

Figure 3. The sequent calculus PUSC$^\perp$

With Kirchner [11], we proved the equivalence of Polarized Sequent Calculus Modulo Theory to a sequent calculus where polarized rewriting rules are applied only on literals, using explicit rules. This calculus, Polarized Unfolding Sequent Calculus, is almost the calculus PUSC$^\perp$ presented in Figure 3. The only difference is that all formulas are put on the left of the sequent in PUSC$^\perp$. We denote by $\Gamma \vdash_{\mathcal{R}}$ the fact that $\Gamma \vdash$ can be proved in PUSC$^\perp$ using the polarized rewriting system $\mathcal{R}$.

We can translate polarized rewriting rules as formulas with selection, and see PUSC$^\perp$ as an instance of LKF$^\perp$. We first consider how to translate formulas of the right-hand side of polarized rewriting rules. We polarize them by choosing positive connectives for $\vee$ and $\wedge$ and, to unchain the introduction of the universal quantifier, we introduce delays. (Let us recall that a delay δ^+ allows to force a formula to be positive.) This gives the translation:

$$\begin{array}{rlrl} |L| = L & \text{when } L \text{ is } \top,\ \bot \text{ or a literal} & & \\ |A \wedge B| = |A| \wedge^+ |B| & & |A \vee B| = |A| \vee^+ |B| & \\ |\exists x.\ A| = \exists x.\ |A| & & |\forall x.\ A| = \forall x.\ \delta^+|A| & \end{array}$$

Definition 5.3.
- Given a negative rewriting rule $P \rightarrow^- A$ where the free variables of P are $x_1, \dots, x_n$, its translation as a formula with selection is $\llbracket P \rightarrow^- A \rrbracket = \forall x_1.\ \dots \forall x_n.\ \underline{\neg P} \vee^- \delta^+|A|$.
- Given a positive rewriting rule $P \rightarrow^+ A$ where the free variables of P are $x_1, \dots, x_n$, its translation as a formula with selection is $\llbracket P \rightarrow^+ A \rrbracket = \forall x_1.\ \dots \forall x_n.\ \underline{P} \vee^- \delta^+|A^\perp|$.

The translation $\llbracket \mathcal{R} \rrbracket$ of a polarized rewriting system $\mathcal{R}$ is the multiset of the translation of its rules.

Definition 5.4. Let $N_1, \dots, N_n$ be a multiset of formulas whose top connective is $\forall$ or $\bot$ or that are literals, and let $P_1, \dots, P_m$ be a multiset of non-literal formulas whose top connective is neither $\forall$ nor $\bot$, then the translation of the PUSC$^\perp$ sequent $N_1, \dots, N_n, P_1, \dots, P_m \vdash$ modulo the rewriting system $\mathcal{R}$ is the LKF$^\perp$ sequent $\llbracket \mathcal{R} \rrbracket, |N_1|, \dots, |N_n| \Uparrow |P_1|, \dots, |P_m| \vdash$.

Theorem 5.5. *$N_1, \dots, N_n, P_1, \dots, P_m \vdash_{\mathcal{R}}$ in PUSC$^\perp$ iff $\llbracket \mathcal{R} \rrbracket, |N_1|, \dots, |N_n| \Uparrow |P_1|, \dots, |P_m| \vdash$ in LKF$^\perp$.*

Proof. Proofs in both calculi correspond almost exactly. $\top\vdash$, $\wedge\vdash$, $\vee\vdash$ and $\exists\vdash$ in $\mathrm{PUSC}^{\bot}$ correspond exactly to $\Uparrow\top\vdash$, $\Uparrow\wedge\vdash$, $\Uparrow\vee\vdash$ and $\Uparrow\exists\vdash$ in $\mathrm{LKF}^{\bot}$, except that if the top connective of the subformulas in the premise(s) is $\forall$ or $\bot$, or if they are literals, they have to be put on the left hand side of $\Uparrow$ using Store. The translation of $\bot\vdash$ corresponds to $\Downarrow\bot\vdash$, except that the latter can only be applied when there is no formula with positive polarity: $\bot\vdash\ \dfrac{}{N_1,\dots,\bot,\dots,N_n \vdash}$ becomes

$$\mathsf{Focus}\ \dfrac{\Downarrow\bot\vdash\ \dfrac{}{\llbracket\mathcal{R}\rrbracket, |N_1|,\dots,\bot,\dots,|N_n| \Downarrow \bot \vdash}}{\llbracket\mathcal{R}\rrbracket, |N_1|,\dots,\bot,\dots,|N_n| \Uparrow \vdash}\ .$$

Similarly, $\forall\vdash$ corresponds to $\Downarrow\forall\vdash$, with the same proviso that there are no formulas with positive polarity:

$$\forall\vdash\ \dfrac{N_1,\dots,N_n,\forall x.\ A,\{t/x\}A \vdash}{N_1,\dots,N_n,\forall x.\ A \vdash}\ \text{becomes}$$

$$\mathsf{Focus}\ \dfrac{\Downarrow\forall\vdash\ \dfrac{\mathsf{Release}\ \dfrac{\llbracket\mathcal{R}\rrbracket, |N_1|,\dots,|N_n|,\forall x.\ \delta^-|A| \Uparrow |\{t/x\}A| \vdash}{\llbracket\mathcal{R}\rrbracket, |N_1|,\dots,|N_n|,\forall x.\ \delta^-|A| \Downarrow \delta^-|\{t/x\}A| \vdash}}{\llbracket\mathcal{R}\rrbracket, |N_1|,\dots,|N_n|,\forall x.\ \delta^-|A| \Downarrow \forall x.\ \delta^-|A| \vdash}}{\llbracket\mathcal{R}\rrbracket, |N_1|,\dots,|N_n|,\forall x.\ \delta^-|A| \Uparrow \vdash}$$

with an extra Store step if the top connective of $\{t/x\}A$ is $\forall$ or $\bot$ or if it is a literal.

For the unfolding rules, if P rewrites positively to A, then there exists a rule $Q \rightarrow^+ B$ and a substitution θ such that $P = \theta Q$ and $A = \theta B$. This rule corresponds to a formula $\llbracket Q \rightarrow^+ B\rrbracket = \overline{\forall x.}\ \underline{Q} \vee^- \delta^+|B^{\bot}|$. Always with the proviso that there is no formula with positive polarity, let $\Gamma = \llbracket\mathcal{R}'\rrbracket, \overline{\forall x.}\ \underline{Q} \vee^- \delta^+|B^{\bot}|, |N_1|,\dots,|N_n|, \neg P$, then $\uparrow^+\vdash$ therefore corresponds to

$$\mathsf{Focus}\ \dfrac{\Downarrow\forall\vdash\ \dfrac{\Downarrow\vee\vdash\ \dfrac{\widehat{\Downarrow}\vdash\ \dfrac{}{\Gamma \Downarrow \underline{P} \vdash} \qquad \mathsf{Release}\ \dfrac{\Gamma \Uparrow |A^{\bot}| \vdash}{\Gamma \Downarrow \delta^+|A^{\bot}| \vdash}}{\Gamma \Downarrow \underline{\theta Q} \vee^- \delta^+|\theta B^{\bot}| \vdash}}{\Gamma \Downarrow \overline{\forall x.}\ \underline{Q} \vee^- \delta^+|B^{\bot}| \vdash}}{\Gamma \Uparrow \vdash}$$

with an extra Store step if the top connective of $|A^{\bot}|$ is $\forall$ or $\bot$, or if it is a literal. Reciprocally, given a sequent $\llbracket\mathcal{R}'\rrbracket, \overline{\forall x.}\ \underline{Q} \vee^- \delta^+|B^{\bot}|, \Gamma \Uparrow\vdash$, if we apply a Focus on $\overline{\forall x.}\ \underline{Q} \vee^- \delta^+|B^{\bot}|$, the derivation is necessarily of the same shape than above, so that there must be a literal $\neg\theta Q$ in Γ. The derivation therefore corresponds to an unfolding of $\neg\theta Q$ into $|\theta B^{\bot}|$.

The case of a negative rewriting is dual.

There remains to be proved that, in $\mathrm{PUSC}^{\bot}$, the rules $\bot\vdash$, $\forall\vdash$ and $\uparrow\vdash$ can be delayed until the other rules are no longer applicable. This can be done by showing that these rules permute with the other ones. Note that this fact can be related with the strategy used in Tamed [4], a tableaux method based on Deduction Modulo Theory, where rules for universal quantifiers and for rewriting are applied when no other rules can be. □

Let us now consider the subcase where the rewriting rules are clausal, according to the terminology of Dowek [20], e.g. they are of the form $P \rightarrow^- C$ or $P \rightarrow^+ \neg C$ for some formula C in clausal normal form. In that case, the resolution method based on Deduction Modulo Theory [21] can be refined into what is called Polarized Resolution Modulo theory [20], whose rules are given in Fig. 4. (A refinement of) Polarized Resolution Modulo theory is actually implemented in the automated theorem prover iProverModulo [7].

$$\mathsf{Resolution}\ \dfrac{P \curlyvee C \qquad \neg Q \curlyvee D}{\sigma(C \curlyvee D)}\ {}^a$$

$$\mathsf{Factoring}\ \dfrac{L \curlyvee K \curlyvee C}{\sigma(L \curlyvee C)}\ \sigma = mgu(L, K)$$

$$\mathsf{Ext.\ Narr.}^-\ \dfrac{P \curlyvee C}{\sigma(D \curlyvee C)}\ {}^a,\ Q \rightarrow^- D$$

$$\mathsf{Ext.\ Narr.}^+\ \dfrac{\neg Q \curlyvee D}{\sigma(C \curlyvee D)}\ {}^a,\ P \rightarrow^+ \neg C$$

$${}^a\sigma = mgu(P, Q)$$

Figure 4. Inference rules of Polarized Resolution Modulo theory

By noting that the translation of the rule $Q \rightarrow^- D$ is $\llbracket Q \rightarrow^- D\rrbracket = \forall x_1.\ \dots\forall x_n.\ \underline{\neg Q} \vee^- \delta^+|D| = \ulcorner \underline{\neg Q} \curlyvee D \urcorner$, we can relate the rule $Q \rightarrow^- D$ with the clause with selection $\underline{\neg Q} \curlyvee D$, which is called a one-way clause by Dowek [20]. Ext. Narr.$^-$ can therefore be seen as an instance of the Resolution with Selection rule:

$$\mathsf{Resolution\ with\ Selection}\ \dfrac{\underline{\neg Q} \curlyvee D \qquad P \curlyvee C}{\sigma(D \curlyvee C)}\ \sigma = mgu(P, Q)\ .$$

Similarly, $P \rightarrow^+ \neg C$ is related to $\underline{P} \curlyvee C$.

Consequently, since $\mathrm{PUSC}^{\bot}$ corresponds to $\mathrm{LKF}^{\bot}$, and Resolution with Input Selection corresponds to Polarized Resolution Modulo theory, Theorem 4.2 leads to a new and more generic proof of the correspondence between $\mathrm{PUSC}^{\bot}$ and Polarized Resolution Modulo theory.

Deduction Modulo Theory is not always complete. This is the case only if the cut rule is admissible in Polarized Sequent Calculus Modulo theory. It holds for some particular theories, e.g. Simple Type Theory [21] and arithmetic [23]. There are more or less powerful techniques that ensures this property [9, 19, 22, 26]. We even proved that any consistent first-order theory can be presented by a rewriting system admitting the cut rule [8]. As presented with Dowek [10] and discussed in the introduction, the fact that completeness is not proved once for all, but needs to be proved for each particular theory, is essential. Indeed, if a theory is presented entirely by rewriting rules, completeness implies the consistency of the theory, since no rule can be applied on the empty set of clauses. Consequently, the proof of the completeness cannot be easier than the proof of consistency of the theory, and, according to Gödel, cannot be proven in the theory itself.

We can go a step further and benefit from focusing to decompose the right-hand side formula after an unfolding has occurred. This leads to what Brauner et al. [5] called Superdeduction. Houtmann [27] studied the links between Superdeduction and focusing, but not with the idea that the rules themselves should be considered as polarized formulas. To link Superdeduction with $\mathrm{LKF}^{\bot}$, we just need to change the translation of rewriting rules in order to ensure that the right-hand side is decomposed as much as possible. This is done by suppressing the positive delay δ^+ and trying to stay in synchronous (i.e. focused) phase by using negative connectives, until we reach a $\exists$ quantifier, after what we try to stay in the asynchronous phase. Note, however, that literals are always given a positive polarization. We introduce the negative translation of a formula:

$$\begin{array}{ll} \multicolumn{2}{c}{\lceil L \rfloor = L \qquad \text{when } L \text{ is } \top, \perp \text{ or a literal}} \\ \lceil A \wedge B \rfloor = \lceil A \rfloor \wedge^- \lceil B \rfloor & \lceil A \vee B \rfloor = \lceil A \rfloor \vee^- \lceil B \rfloor \\ \lceil \exists x.\ A \rfloor = \exists x.\ |A| & \lceil \forall x.\ A \rfloor = \forall x.\ \lceil A \rfloor \end{array}$$

and the translation of rewrite rules becomes:

$$\llbracket P \rightarrow^- A \rrbracket = \overline{\forall x.}\ \underline{\neg P} \vee^- \lceil A \rfloor$$
$$\llbracket P \rightarrow^+ A \rrbracket = \overline{\forall x.}\ \underline{P} \vee^- \lceil A^\perp \rfloor$$

The synthetic rules given by the translation of rewriting rules correspond exactly to the superrules of Superdeduction.

Note that the same kind of encodings can be used to show that Definitional reflection, as defined by Schroeder-Heister [32], can be seen as an instance of LKF$^\perp$.

5.3 Beyond Deduction Modulo Theory

Example 5.6. Let us recall the set of clauses of the introduction:

$$\underline{\neg X \in \mathcal{P}(Y)} \curlyvee \underline{\neg Z \in X} \curlyvee Z \in Y \qquad (1)$$
$$\underline{X \in \mathcal{P}(Y) \curlyvee d(X,Y) \in X} \qquad (2)$$
$$\underline{X \in \mathcal{P}(Y)} \curlyvee \underline{\neg d(X,Y) \in Y} \qquad (3)$$

Note that this example is not covered by Ordered Resolution with Selection, at least not if a simplification ordering is used, because we cannot have $X \in \mathcal{P}(Y) \succ \delta(X,Y) \in X$ since with $\theta = \{X \mapsto \mathcal{P}(Z); Y \mapsto Z\}$ their instances are ordered in the wrong direction: $\mathcal{P}(Z) \in \mathcal{P}(Z) \prec \delta(\mathcal{P}(Z), Z) \in \mathcal{P}(Z)$.

The synthetic rules of the example in the introduction correspond to the derivations when one of the clauses is focused. For instance, if we consider the clause (1), in a context Γ containing this clause, a proof putting the focus on $\ulcorner(1)\urcorner$ necessarily is of the following shape:

$$\text{Focus}\ \dfrac{\Downarrow\forall\vdash\ \dfrac{\Downarrow\vee\vdash\ \dfrac{\widehat{\Downarrow}\vdash\ \dfrac{}{\Gamma \Downarrow \underline{\neg u \in \mathcal{P}(v)} \vdash} \qquad \widehat{\Downarrow}\vdash\ \dfrac{}{\Gamma \Downarrow \underline{\neg t \in u} \vdash} \qquad \text{Release}\ \dfrac{\text{Store}\ \dfrac{\Gamma, t \in v \Uparrow \vdash}{\Gamma \Uparrow t \in v \vdash}}{\Gamma \Downarrow t \in v \vdash}}{\Gamma \Downarrow \underline{\neg u \in \mathcal{P}(v)} \vee^- \underline{\neg t \in u} \vee^- t \in v \vdash}}{\Gamma \Downarrow \forall X\ Y\ Z.\ \underline{\neg X \in \mathcal{P}(Y)} \vee^- \underline{\neg Z \in X} \vee^- Z \in Y \vdash}}{\Gamma \Uparrow \vdash}$$

where t, u, v are arbitrary terms, and where, to be able to close the left and middle branch, $u \in \mathcal{P}(v)$ and $t \in u$ must belong to Γ. So Γ is in fact $\forall X\ Y\ Z.\ \underline{\neg X \in \mathcal{P}(Y)} \vee^+ \underline{\neg Z \in X} \vee^+ Z \in Y, \Delta, u \in \mathcal{P}(v), t \in u$ for some Δ, and the axiom $\forall X\ Y\ Z.\ \underline{\neg X \in \mathcal{P}(Y)} \vee^+ \underline{\neg Z \in X} \vee^+ Z \in Y$ can be replaced by the synthetic rule:

$$(1)\ \dfrac{\Delta, u \in \mathcal{P}(v), t \in u, t \in v \Uparrow \vdash}{\Delta, u \in \mathcal{P}(v), t \in u \Uparrow \vdash}\ .$$

The computation of the other synthetic rules is left as an exercise for the reader.

On the resolution side, if we consider the ground instances of Resolution with Selection, we have:

$$\text{R.w.S.}\ \dfrac{\underline{\neg X \in \mathcal{P}(Y)} \curlyvee \underline{\neg Z \in X} \curlyvee Z \in Y \quad u \in \mathcal{P}(v) \curlyvee C \quad t \in u \curlyvee D}{t \in v \curlyvee C \curlyvee D}$$

$$\text{R.w.S.}\ \dfrac{\underline{X \in \mathcal{P}(Y)} \curlyvee d(X,Y) \in X \qquad \neg u \in \mathcal{P}(v) \curlyvee C}{d(u,v) \in u \curlyvee C}$$

$$\text{R.w.S.}\ \dfrac{\underline{X \in \mathcal{P}(Y)} \curlyvee \underline{\neg d(X,Y) \in Y} \quad \neg u \in \mathcal{P}(v) \curlyvee C \quad d(u,v) \in v \curlyvee D}{C \curlyvee D}$$

hence the derived rules given in the introduction, where the clauses of the theory are not mentioned.

The question that remains is how we can prove the completeness of such a selection. We can in fact reduce it to the completeness for several rewriting systems in Deduction Modulo Theory.

Definition 5.7 (Singleton subselection)**.** Given a selection function $\mathcal{S}$, the selection function $\mathcal{S}_1$ is a singleton subselection of $\mathcal{S}$ if

- $\mathcal{S}_1(C) \subseteq \mathcal{S}(C)$ for all C
- if $\mathcal{S}(C) \neq \emptyset$ then $\mathrm{card}(\mathcal{S}_1(C)) = 1$

Example 5.8.

$$\neg X \in \mathcal{P}(Y) \curlyvee \underline{\neg Z \in X} \curlyvee Z \in Y$$
$$\underline{X \in \mathcal{P}(Y)} \curlyvee d(X,Y) \in X$$
$$\underline{X \in \mathcal{P}(Y)} \curlyvee \neg d(X,Y) \in Y$$

is a singleton subselection of Example 5.6.

Theorem 5.9. *Resolution with input selection $\mathcal{S}$ is complete iff for all singleton subselection $\mathcal{S}_1$ of $\mathcal{S}$, Resolution with input selection $\mathcal{S}_1$ is complete*

Proof. See Appendix. □

In particular, to prove completeness if we have a clause $\underline{L_1} \curlyvee \ldots \curlyvee \underline{L_n} \curlyvee K_1 \curlyvee \ldots \curlyvee K_m$, one can prove completeness of the n singleton subselection

$$\underline{L_1} \curlyvee \ldots \curlyvee L_n \curlyvee K_1 \curlyvee \ldots \curlyvee K_m$$
$$\vdots$$
$$L_1 \curlyvee \ldots \curlyvee \underline{L_n} \curlyvee K_1 \curlyvee \ldots \curlyvee K_m$$

which correspond, according to Section 5.2, to the n rewriting systems

$$L_1 \rightarrow^\pm (\neg)L_2 \vee \cdots \vee L_n \vee K_1 \vee \cdots \vee K_m$$
$$\vdots$$
$$L_n \rightarrow^\pm (\neg)L_1 \vee \cdots \vee L_{n-1} \vee K_1 \vee \cdots \vee K_m$$

in Deduction Modulo Theory.

Conclusion and Further Work

We generalized focusing and resolution with selection, proved that they correspond, and showed how known calculi are instances of this framework, namely ordinary focusing, hyperresolution, Deduction Modulo Theory and Superdeduction. We also showed how to reduce completeness of this framework to several completeness proofs in Deduction Modulo Theory. We can therefore reuse the various techniques for proving completeness in Deduction Modulo Theory [9, 19, 22, 26] in our framework. These notable results raise the following new areas of investigations.

First, we need to study how to apply selection also in the generated clauses. This should allow us to cover the cases of Ordered Resolution with Selection and of Semantic Resolution in the first-order case. Dually, in the sequent calculus part, this would correspond to the possibility to dynamically add selection in formulas of subderivations. This could probably be linked with the work of Deplagne and Kirchner [17] where rewrite rules corresponding to induction hypotheses are dynamically added in the rewriting system of a sequent calculus for Deduction Modulo Theory. Note that we already have one direction, namely from Resolution with Input Selection to LKF$^{\perp}$, since Lemmas 4.7, 4.9, and 4.10 does not assume anything on the generated clauses; except, for Factoring, that it selects only instances of literals that were already selected. The converse direction would require a meta-theorem of completeness, since obviously it is not complete for all possible dynamic choice of selection.

Since focusing is defined not only for classical first-order logic but also for linear, intuitionistic, modal logics, the work in this paper could serve as a starting point to study how to get automated proof search methods for these logics with a selection mechanism.

Another worthwhile point is how equality should be handled in our framework. In particular, it would be interesting to see how paramodulation calculi, in particular superposition, can be embedded into a sequent calculus.

Finally, it would be worth investigating whether completeness proofs based on model construction, such as semantic completeness proofs of tableaux (related to sequent calculus), and completeness proof of superposition [2], can be related in our framework.

References

[1] Jean-Marc Andreoli. 1992. Logic Programming with Focusing Proofs in Linear Logic. *Journal of Logic and Computation* 2, 3 (1992), 297–347.

[2] L. Bachmair and H. Ganzinger. 1994. Rewrite-based equational theorem proving with selection and simplification. *Journal of Logic and Computation* 4, 3 (1994), 1–31.

[3] Leo Bachmair and Harald Ganzinger. 2001. Resolution Theorem Proving. In *Handbook of Automated Reasoning*, John Alan Robinson and Andrei Voronkov (Eds.). Elsevier and MIT Press, 19–99.

[4] Richard Bonichon and Olivier Hermant. 2006. A Semantic Completeness Proof for TaMed. In *LPAR (LNCS)*, Miki Hermann and Andrei Voronkov (Eds.), Vol. 4246. Springer, 167–181.

[5] Paul Brauner, Clément Houtmann, and Claude Kirchner. 2007. Principle of superdeduction. In *LICS*, Luke Ong (Ed.). 41–50.

[6] Guillaume Burel. 2010. Embedding Deduction Modulo into a Prover. In *CSL (LNCS)*, Anuj Dawar and Helmut Veith (Eds.), Vol. 6247. Springer, 155–169.

[7] Guillaume Burel. 2011. Experimenting with Deduction Modulo. In *CADE (LNCS)*, Viorica Sofronie-Stokkermans and Nikolaj Bjørner (Eds.), Vol. 6803. Springer, 162–176.

[8] Guillaume Burel. 2013. From Axioms to Rewriting Rules. (2013). Available on author's web page.

[9] Guillaume Burel. 2014. Cut Admissibility by Saturation. In *RTA-TLCA (LNCS)*, Gilles Dowek (Ed.), Vol. 8560. Springer, 124–138.

[10] Guillaume Burel and Gilles Dowek. 2009. How can we prove that a proof search method is not an instance of another?. In *LFMTP (ACM International Conference Proceeding Series)*. ACM, 84–87.

[11] Guillaume Burel and Claude Kirchner. 2010. Regaining Cut Admissibility in Deduction Modulo using Abstract Completion. *Information and Computation* 208, 2 (2010), 140–164.

[12] Samuel R. Buss (Ed.). 1998. *Handbook of proof theory*. Elsevier, Amsterdam.

[13] Kaustuv Chaudhuri and Frank Pfenning. 2005. A Focusing Inverse Method Theorem Prover for First-Order Linear Logic. In *CADE (LNCS)*, Robert Nieuwenhuis (Ed.), Vol. 3632. Springer, 69–83.

[14] Kaustuv Chaudhuri, Frank Pfenning, and Greg Price. 2008. A Logical Characterization of Forward and Backward Chaining in the Inverse Method. *Journal of Automated Reasoning* 40, 2-3 (2008), 133–177.

[15] Zakaria Chihani, Tomer Libal, and Giselle Reis. 2015. The Proof Certifier Checkers. In *TABLEAUX (LNCS)*, Vol. 9323. Springer, Wroclaw, Poland, 201–210.

[16] Zakaria Chihani, Dale Miller, and Fabien Renaud. 2013. Foundational Proof Certificates in First-Order Logic. In *CADE (LNCS)*, Maria Paola Bonacina (Ed.), Vol. 7898. Springer, 162–177.

[17] Eric Deplagne and Claude Kirchner. 2004. *Induction as Deduction Modulo*. Rapport de recherche. LORIA. http://www.loria.fr/publications/2004/A04-R-468/A04-R-468.ps

[18] Gilles Dowek. 2002. What Is a Theory?. In *STACS (LNCS)*, Helmut Alt and Afonso Ferreira (Eds.), Vol. 2285. Springer, 50–64.

[19] Gilles Dowek. 2006. Truth Values Algebras and Proof Normalization. In *TYPES (LNCS)*, Thorsten Altenkirch and Conor McBride (Eds.), Vol. 4502. Springer, 110–124.

[20] Gilles Dowek. 2010. Polarized Resolution Modulo. In *IFIP TCS (IFIP AICT)*, Cristian S. Calude and Vladimiro Sassone (Eds.), Vol. 323. Springer, 182–196.

[21] Gilles Dowek, Thérèse Hardin, and Claude Kirchner. 2003. Theorem Proving Modulo. *Journal of Automated Reasoning* 31, 1 (2003), 33–72.

[22] Gilles Dowek and Benjamin Werner. 2003. Proof Normalization Modulo. *The Journal of Symbolic Logic* 68, 4 (2003), 1289–1316.

[23] Gilles Dowek and Benjamin Werner. 2005. Arithmetic as a Theory Modulo. In *RTA (LNCS)*, Jürgen Giesl (Ed.), Vol. 3467. Springer, 423–437.

[24] Mahfuza Farooque, Stéphane Graham-Lengrand, and Assia Mahboubi. 2013. A bisimulation between DPLL(T) and a proof-search strategy for the focused sequent calculus. In *LFMTP*, Alberto Momigliano, Brigitte Pientka, and Randy Pollack (Eds.). ACM, 3–14.

[25] Jean Goubault-Larrecq. 2002. *A Note on the Completeness of Certain Refinements of Resolution*. Research Report LSV-02-8. Laboratoire Spécification et Vérification, ENS Cachan, France.

[26] Olivier Hermant. 2005. *Méthodes Sémantiques en Déduction Modulo*. Ph.D. Dissertation. École Polytechnique.

[27] Clément Houtmann. 2008. Axiom directed Focusing. In *Types for Proofs and Programs (LNCS)*, Stephano Berardi, Ferruccio Damiani, and Ugo de'Liguoro (Eds.), Vol. 5497. Springer, 169–185.

[28] Chuck Liang and Dale Miller. 2009. Focusing and polarization in linear, intuitionistic, and classical logics. *Theoretical Computer Science* 410, 46 (2009), 4747–4768. Abstract Interpretation and Logic Programming: In honor of professor Giorgio Levi.

[29] Sean McLaughlin and Frank Pfenning. 2008. Imogen: Focusing the Polarized Inverse Method for Intuitionistic Propositional Logic. In *LPAR (Lecture Notes in Computer Science)*, Iliano Cervesato, Helmut Veith, and Andrei Voronkov (Eds.), Vol. 5330. Springer, 174–181.

[30] Dale Miller and Marco Volpe. 2015. Focused Labeled Proof Systems for Modal Logic. In *LPAR (LNCS)*, Martin Davis, Ansgar Fehnker, Annabelle McIver, and Andrei Voronkov (Eds.), Vol. 9450. Springer, 266–280.

[31] J. A. Robinson. 1965. Automatic Deduction with Hyper-Resolution. *International Journal of Computer Mathematics* 1 (1965), 227–234.

[32] Peter Schroeder-Heister. 1990. Cut Elimination for Logics with Definitional Reflection.. In *Nonclassical Logics and Information Processing (LNCS)*, Vol. 619. Springer, 146–171.

[33] James R. Slagle. 1967. Automatic Theorem Proving With Renamable and Semantic Resolution. *J. ACM* 14, 4 (1967), 687–697.

[34] Larry Wos, George A. Robinson, and Daniel F. Carson. 1965. Efficiency and Completeness of the Set of Support Strategy in Theorem Proving. *J. ACM* 12, 4 (1965), 536–541.

A Appendix

Proofs of the lemmas

Proof of Lemma 4.3. By induction on the derivation length of $\Gamma, C_1, \ldots, C_n \rightsquigarrow^* \square$, generalizing on Γ.

The base case is when $\square \in \Gamma$. Then, trivially $\Gamma, C_1 \curlyvee D, \ldots, C_n \curlyvee D \rightsquigarrow^* \square$.

For the inductive case, suppose that there exists C_{n+1} such that $\Gamma, C_1, \ldots, C_n \rightsquigarrow C_{n+1}$ and $\Gamma, C_1, \ldots, C_n, C_{n+1} \rightsquigarrow^* \square$.

There are two cases:

- C_{n+1} is derived using other clauses than one of the C_i. We therefore have $\Gamma, C_1 \curlyvee D, \ldots, C_n \curlyvee D \rightsquigarrow C_{n+1}$. We can apply the induction hypothesis on $\Gamma, C_1, \ldots, C_n, C_{n+1} \rightsquigarrow^* \square$, which can be viewed as $\Gamma, C_{n+1}, C_1, \ldots, C_n \rightsquigarrow^* \square$, since by weakening we have $\Gamma, C_{n+1}, D \rightsquigarrow^* \square$. We obtain $\Gamma, C_{n+1}, C_1 \curlyvee D, \ldots, C_n \curlyvee D \rightsquigarrow^* \square$. By definition of $\rightsquigarrow^*$ we therefore have $\Gamma, C_1 \curlyvee D, \ldots, C_n \curlyvee D \rightsquigarrow^* \square$.
- At least one of the parents of C_{n+1} is some C_i. Since no literal is selected in the C_i, they can only be side clauses of Resolution with Selection, or any clause in Resolution or Factoring. We can therefore derive $C_{n+1} \curlyvee D$ from $\Gamma, C_1 \curlyvee D, \ldots, C_n \curlyvee D$ with the same inference rule that produced C_n. We can apply the induction hypothesis on $\Gamma, C_1, \ldots, C_n, C_{n+1} \rightsquigarrow^* \square$ which gives us $\Gamma, C_1 \curlyvee D, \ldots, C_n \curlyvee D, C \curlyvee D \rightsquigarrow^* \square$. Hence $\Gamma, C_1 \curlyvee D, \ldots, C_n \curlyvee D \rightsquigarrow^* \square$. □

Proof of Lemma 4.5. By induction on the derivation length of $\Gamma, \theta C \rightsquigarrow^* \square$. As in the previous proof, the only interesting case is when the clause θC is used in the first step $\Gamma, \theta C \rightsquigarrow D$ of the derivation. If the first step $\Gamma, \theta C \rightsquigarrow D$ is Factoring, then $D = \sigma\theta C$ where σ is the most general unifier of two literals in θC. We can apply the induction hypothesis using $\sigma\theta$ instead of σ.

Otherwise, if the first step is Resolution with Selection or Resolution. Let θC be $L \curlyvee \theta C'$ where L is the literal of θC used in that step. Let C be $L_1 \curlyvee \cdots \curlyvee L_n \curlyvee C'$ where the L_i are exactly the literals of C such that $\theta L_i = L$. The step produces therefore a clause $D = \sigma(\theta C' \curlyvee D')$ where σ is the most general unifier of a unification problem $L \stackrel{?}{=} L', Prob$. Let $\omega = mgu(L_1, \ldots, L_n)$, then $\theta = \theta'\omega$. First, from C one can derive $\omega L_1 \curlyvee \omega C'$ by repetitively applying Factoring to C. We have $\sigma L = \sigma L'$, hence $\sigma\theta'\omega L_1 = \sigma L'$. Since we rename variables in the premises of the resolution rules, we can assume that θ' does not affect the variables of L'. Consequently, $\sigma\theta'\omega L_1 = \sigma\theta' L'$. Thus, $\sigma\theta$ is a solution of the unification problem $\omega L_1 \stackrel{?}{=} L', Prob$. Let μ be its most general solution. Therefore, there exists κ such that $\sigma\theta' = \kappa\mu$.

From $\omega L_1 \curlyvee \omega C'$ and the same other clauses as in the step $\Gamma, \theta C \rightsquigarrow D$, we can therefore derive $\mu(\omega C' \curlyvee D')$. Once again, the variables of D', coming from the other clauses, can be assumed to be distinct of those of C, therefore $D' = \theta' D'$. We have $\sigma(\theta C' \curlyvee D') = \sigma(\theta'\omega C' \curlyvee D') = \sigma\theta'(\omega C' \curlyvee D') = \kappa\mu(\omega C' \curlyvee D)$. We can therefore apply the induction hypothesis, using κ on $\mu(\omega C' \curlyvee D)$ instead of θ on C. □

Proof of Lemma 4.7. By induction on the proof $\Gamma, \ulcorner C \urcorner, \ulcorner \sigma C \urcorner \Uparrow\vdash$. If the proof does not begin by focusing on $\ulcorner \sigma C \urcorner$, this is a simple application of the induction hypothesis (considering coarse grain proof steps consisting of an alternation of a synchronous and an asynchronous phases). Otherwise, let σC be $\underline{K_1} \curlyvee \cdots \curlyvee \underline{K_n} \curlyvee L_1 \curlyvee \cdots \curlyvee L_m$. The proof begins with

$$\mathsf{Focus}\ \dfrac{\Downarrow\forall\vdash\ \dfrac{\Downarrow\vee\vdash\ \dfrac{\cdots\quad \widehat{\Downarrow}\vdash\ \dfrac{}{\Gamma' \Downarrow \underline{\theta K_j} \vdash}\quad \cdots \qquad \mathsf{Release}\ \dfrac{\Uparrow\vee\vdash\ \dfrac{\cdots\quad \mathsf{Store}\ \dfrac{\Gamma', \theta L_k \Uparrow \vdash}{\Gamma' \Uparrow \theta L_k \vdash}\quad \cdots}{\Gamma' \Uparrow \theta(L_1 \vee^+ \cdots \vee^+ L_m) \vdash}}{\Gamma' \Downarrow \theta(L_1 \vee^+ \cdots \vee^+ L_m) \vdash}}{\Gamma' \Downarrow \theta(\underline{K_1} \vee^- \cdots \vee^- \underline{K_n} \vee^- (L_1 \vee^+ \cdots \vee^+ L_m)) \vdash}}{\Gamma' \Downarrow \overline{\forall x}.\underline{K_1} \vee^- \cdots \vee^- \underline{K_n} \vee^- (L_1 \vee^+ \cdots \vee^+ L_m) \vdash}}{\Gamma' \Uparrow \vdash}$$

where $\Gamma' = \Gamma, \ulcorner C \urcorner, \ulcorner \sigma C \urcorner$ and for all $1 \leq j \leq n$ the literal $\theta K_j{}^\perp$ is in Γ.

Let C be
$\underline{K_1^1} \curlyvee \cdots \underline{K_1^{k_1}} \curlyvee \cdots \underline{K_n^1} \curlyvee \cdots \underline{K_n^{k_n}} \curlyvee L_1^1 \curlyvee \cdots L_1^{l_1} \curlyvee \cdots L_m^1 \curlyvee \cdots L_m^{l_m}$
where $\sigma K_i^j = K_i$ and $\sigma L_i^j = L_i$ for all i, j. By hypothesis, the literals selected in C are among the K_i^j. For all i, j, one can either build a proof $\widehat{\Downarrow}\vdash\ \dfrac{}{\Gamma, \ulcorner C \urcorner \Downarrow \underline{\theta\sigma K_i^j} \vdash}$ if it is selected, or $\mathsf{Store}\ \dfrac{\widehat{\Uparrow}\vdash\ \dfrac{}{\Gamma, \ulcorner C \urcorner, \theta\sigma K_i^j \Uparrow \vdash}}{\Gamma, \ulcorner C \urcorner \Uparrow \theta\sigma K_i^j \vdash}$ if it is not.

We apply the induction hypothesis on $\Gamma', \theta L_k \Uparrow\vdash$, hence we have proofs $\Gamma, \ulcorner C \urcorner, \theta L_k \Uparrow\vdash$ for all k.

We can therefore build the proof in Figure 5. □

Proof of Lemma 4.9. By induction on the proof $\Gamma, \ulcorner L \curlyvee C \urcorner, \ulcorner L'^\perp \curlyvee D \urcorner, \ulcorner \sigma(C \curlyvee D) \urcorner \Uparrow\vdash$.
Let Γ' be $\Gamma, \ulcorner L \curlyvee C \urcorner, \ulcorner L'^\perp \curlyvee D \urcorner$ and Γ'' be $\Gamma', \ulcorner \sigma(C \curlyvee D) \urcorner$. If the proof does not begin by focusing on $\ulcorner \sigma(C \curlyvee D) \urcorner$, this is a simple application of the induction hypothesis. Otherwise, let $\sigma(C \curlyvee D)$ be $\underline{I_1} \curlyvee \cdots \curlyvee \underline{I_n} \curlyvee J_1 \curlyvee \cdots \curlyvee J_m$. The proof begins with

$$\mathsf{Focus}\ \dfrac{\Downarrow\forall\vdash\ \dfrac{\Downarrow\vee\vdash\ \dfrac{\cdots\quad \widehat{\Downarrow}\vdash\ \dfrac{}{\Gamma'' \Downarrow \underline{\theta I_j} \vdash}\quad \cdots \qquad \mathsf{Release}\ \dfrac{\Uparrow\vee\vdash\ \dfrac{\cdots\quad \mathsf{Store}\ \dfrac{\Gamma'', \theta J_k \Uparrow \vdash}{\Gamma'' \Uparrow \theta J_k \vdash}\quad \cdots}{\Gamma'' \Uparrow \theta(J_1 \vee^+ \cdots \vee^+ J_m) \vdash}}{\Gamma'' \Downarrow \theta(J_1 \vee^+ \cdots \vee^+ J_m) \vdash}}{\Gamma'' \Downarrow \theta(\underline{I_1} \vee^- \cdots \vee^- \underline{I_n} \vee^- (J_1 \vee^+ \cdots \vee^+ J_m)) \vdash}}{\Gamma'' \Downarrow \overline{\forall x}.\underline{I_1} \vee^- \cdots \vee^- \underline{I_n} \vee^- (J_1 \vee^+ \cdots \vee^+ J_m) \vdash}}{\Gamma'' \Uparrow \vdash}$$

where, to be able to close the left branches, for all $1 \leq j \leq n$ the literal $\theta I_j{}^\perp$ is in Γ'.

We know that C is $\cdots \curlyvee I_i^1 \curlyvee \cdots I_i^{k_i} \curlyvee \cdots \curlyvee J_j^1 \curlyvee \cdots J_j^{l_j} \curlyvee \cdots$ where i ranges over a subset of $\{1, \ldots, n\}$ and j over a subset of $\{1, \ldots, m\}$, and $\sigma I_i^x = I_i$ and $\sigma J_j^y = J_j$ for all x, y. Likewise, D is $\cdots \curlyvee I_i'^1 \curlyvee \cdots I_i'^{k_i} \curlyvee \cdots \curlyvee J_j'^1 \curlyvee \cdots J_j'^{l_j} \curlyvee \cdots$ where i ranges over a subset of $\{1, \ldots, n\}$ and j over a subset of $\{1, \ldots, m\}$, and $\sigma I_i'^x = I_i$ and $\sigma J_j'^y = J_j$ for all x, y.

$$
\dfrac{
\dfrac{\overline{\Gamma, \ulcorner C \urcorner \Downarrow \underline{\theta\sigma K_i^j} \vdash}\,\widehat{\Downarrow}\vdash \qquad \cdots \qquad
\dfrac{\dfrac{\dfrac{\overline{\Gamma, \ulcorner C \urcorner, \theta\sigma K_i^j \Uparrow \vdash}\,\widehat{\Uparrow}\vdash}{\Gamma, \ulcorner C \urcorner \Uparrow \theta\sigma K_i^j \vdash}\,\text{Store} \qquad \cdots \qquad \Gamma, \ulcorner C \urcorner, \theta L_i \Uparrow \vdash}{\Gamma, \ulcorner C \urcorner \Uparrow \cdots \vee^+ \theta\sigma K_l^m \vee^+ \cdots \vee^+ \theta\sigma L_i^j \vee^+ \cdots \vdash}\,\Uparrow\vee\vdash}{\Gamma, \ulcorner C \urcorner \Downarrow \cdots \vee^+ \theta\sigma K_l^m \vee^+ \cdots \vee^+ \theta\sigma L_i^j \vee^+ \cdots \vdash}\,\text{Release}}
{\dfrac{\Gamma, \ulcorner C \urcorner \Downarrow \cdots \vee^- \underline{\theta\sigma K_i^j} \vee^- \cdots \vee^- (\cdots \vee^+ \theta\sigma K_l^m \vee^+ \cdots \vee^+ \theta\sigma L_i^j \vee^+ \cdots) \vdash}{\overline{\Gamma, \ulcorner C \urcorner \Downarrow \overline{\forall x}. \cdots \vee^- \underline{K_i^j} \vee^- \cdots \vee^- (\cdots \vee^+ K_l^m \vee^+ \cdots \vee^+ L_i^j \vee^+ \cdots) \vdash}}\,\Downarrow\forall\vdash}\,\Downarrow\vee\vdash
}{\Gamma, \ulcorner C \urcorner \Uparrow \vdash}\,\text{Focus}
$$

Figure 5. Proof for Lemma 4.7

We apply the induction hypothesis on $\Gamma'', \theta J_k \Uparrow\vdash$, hence we have proof $\Gamma, \ulcorner L \curlyvee C \urcorner, \ulcorner L'^{\perp} \curlyvee D \urcorner, \theta J_k \Uparrow\vdash$.

We can build the following proof of $\Gamma' \vdash$: first, we focus on $\ulcorner L \curlyvee C \urcorner$ to get the derivation (a) of Figure 6. In this derivation, all right branches are closed by induction hypothesis. On the left branch, we focus on $\ulcorner L'^{\perp} \curlyvee D \urcorner$. Let Γ''' be $\Gamma', \theta\sigma L$. We get the derivation (b) of Figure 6. The left branch can be closed because $\theta\sigma L = \theta\sigma L'$. The right branches are closed by induction hypothesis. □

Proof of Lemma 4.10. By induction on the proof $\Gamma, \ulcorner \underline{K_1} \curlyvee ... \curlyvee \underline{K_n} \curlyvee C \urcorner, ..., \ulcorner K_i'^{\perp} \curlyvee D_i \urcorner, ..., \ulcorner \sigma(C \curlyvee D_1 \curlyvee ... \curlyvee D_n) \urcorner \Uparrow\vdash$. We follow the same idea as in the proofs of the two precedent lemmas. If the proof does not begin by focusing on $\ulcorner \sigma(C \curlyvee D_1 \curlyvee \cdots \curlyvee D_n) \urcorner$, this is a simple application of the induction hypothesis. Otherwise, let Γ' be $\Gamma, \ulcorner \underline{K_1} \curlyvee \ldots \curlyvee \underline{K_n} \curlyvee C \urcorner, \ldots \ulcorner K_i'^{\perp} \curlyvee D_i \urcorner, \ldots$ and Γ'' be $\Gamma', \ulcorner \sigma(C \curlyvee D_1 \curlyvee \cdots \curlyvee D_n) \urcorner$. Focusing leads us either to sequents $\Gamma'' \Downarrow \underline{\theta I_j} \vdash$, with $\theta {I_j}^{\perp}$ in Γ, or to sequents $\Gamma'', \theta J_k \Uparrow\vdash$ upon which one can apply the induction hypothesis. Let us remark that for each literal L of C or D_i, $\theta\sigma L$ is either one of θI_j or one of θJ_k. Therefore, we know how to close proofs of $\Gamma' \Uparrow \theta\sigma L \vdash$ for each, either by induction hypothesis or using $\theta {I_j}^{\perp}$ in Γ.

To build the proof of $\Gamma' \vdash$, we first focus on $\ulcorner K_1'^{\perp} \curlyvee D_1 \urcorner$, instantiating the variables using the substitution $\theta\sigma$. As explain above, we know how to close the branches coming from D_1, it remains the branch $\Gamma', \theta\sigma K_1'^{\perp} \Uparrow\vdash$.

We do the same, focusing on $\ulcorner K_2'^{\perp} \curlyvee D_2 \urcorner$ then ... then $\ulcorner K_n'^{\perp} \curlyvee D_n \urcorner$ and the remaining branch is $\Gamma', \theta\sigma K_1'^{\perp}, \ldots \theta\sigma K_n'^{\perp} \Uparrow\vdash$.

We can close the proof by focusing on $\ulcorner \underline{K_1} \curlyvee \ldots \curlyvee \underline{K_n} \curlyvee C \urcorner$. Branches coming from C can be closed as before, and the other branches are closed by $\overline{\Gamma', \theta\sigma K_1'^{\perp}, \ldots \theta\sigma K_n'^{\perp} \Downarrow \underline{\theta\sigma K_i} \vdash}\,\widehat{\Downarrow}\vdash$ since $\theta\sigma K_i' = \theta\sigma K_i$ for all i. □

Proof sketch of Theorem 5.9. Showing that LKF$^{\perp}$ is complete amounts to proving that the cut rule

$$\dfrac{\Gamma, A \Uparrow \vdash \qquad \Gamma, A^{\perp} \Uparrow \vdash}{\Gamma \Uparrow \vdash}\,\underset{\smile}{\leftharpoonup}$$

is admissible.

Using the same techniques as in [11], we can try to eliminate cuts using structural cut elimination à la Pfenning. The only problematic case is when we cut around a literal that is used in focused instance in both branches:

$$\dfrac{\dfrac{\dfrac{\overline{\Gamma', A \Downarrow \underline{A^{\perp}} \vdash}\,\widehat{\Downarrow}\vdash}{\vdots}}{\dfrac{\Gamma, A \Downarrow E[\underline{A^{\perp}}] \vdash}{\Gamma, A \Uparrow \vdash}\,\text{Focus}} \qquad \dfrac{\dfrac{\overline{\Gamma'', A^{\perp} \Downarrow \underline{A} \vdash}\,\widehat{\Downarrow}\vdash}{\vdots}}{\dfrac{\Gamma, A^{\perp} \Downarrow F[\underline{A}] \vdash}{\Gamma, A^{\perp} \Uparrow \vdash}\,\text{Focus}}}{\Gamma \Uparrow \vdash}\,\underset{\smile}{\leftharpoonup}$$

But, since LKF$^{\perp}$is complete for all singleton subselection, it is complete if one selects only $A^{\perp}$ in $E[\underline{A^{\perp}}]$ and only A in $F[\underline{A}]$. Consequently, we know how to eliminate the cut above. □

$$\text{(a)}\quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma', \theta\sigma L \Uparrow \vdash}{\Gamma' \Uparrow \theta\sigma L \vdash}\,\text{Store} \quad \cdots \quad \dfrac{\dfrac{}{\Gamma', \theta I_i \Uparrow \vdash}\,\widehat{\Uparrow}\vdash}{\Gamma' \Uparrow \theta I_i \vdash}\,\text{Store} \quad \cdots \quad \dfrac{\Gamma', \theta J_j \Uparrow \vdash}{\Gamma' \Uparrow \theta J_j \vdash}\,\text{Store} \quad \cdots}{\Gamma' \Uparrow \theta\sigma(L \vee^+ \cdots \vee^+ I_i^1 \vee^+ \cdots I_i^{k_i} \vee^+ \cdots \vee^+ J_j^1 \vee^+ \cdots J_j^{l_j} \vee^+ \cdots) \vdash}\,\Uparrow\vee\vdash}{\Gamma' \Downarrow \theta\sigma(L \vee^+ \cdots \vee^+ I_i^1 \vee^+ \cdots I_i^{k_i} \vee^+ \cdots \vee^+ J_j^1 \vee^+ \cdots J_j^{l_j} \vee^+ \cdots) \vdash}\,\text{Release}}{\Gamma' \Downarrow \overline{\forall x}.\ L \vee^+ \cdots \vee^+ I_i^1 \vee^+ \cdots I_i^{k_i} \vee^+ \cdots \vee^+ J_j^1 \vee^+ \cdots J_j^{l_j} \vee^+ \cdots \vdash}\,\Downarrow\forall\vdash}{\Gamma' \Uparrow \vdash}\,\text{Focus}$$

$$\text{(b)}\quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma''', \theta\sigma L'^{\perp} \Uparrow \vdash}\,\widehat{\Uparrow}\vdash}{\Gamma''' \Uparrow \theta\sigma L'^{\perp} \vdash}\,\text{Store} \quad \cdots \quad \dfrac{\dfrac{}{\Gamma''', \theta I_i \Uparrow \vdash}\,\widehat{\Uparrow}\vdash}{\Gamma''' \Uparrow \theta I_i \vdash}\,\text{Store} \quad \cdots \quad \dfrac{\Gamma''', \theta J_j \Uparrow \vdash}{\Gamma''' \Uparrow \theta J_j \vdash}\,\text{Store} \quad \cdots}{\Gamma''' \Uparrow \theta\sigma(L'^{\perp} \vee^+ \cdots \vee^+ I_i'^1 \vee^+ \cdots I_i'^{k_i} \vee^+ \cdots \vee^+ J_j'^1 \vee^+ \cdots J_j'^{l_j} \vee^+ \cdots) \vdash}\,\Uparrow\vee\vdash}{\Gamma''' \Downarrow \theta\sigma(L'^{\perp} \vee^+ \cdots \vee^+ I_i'^1 \vee^+ \cdots I_i'^{k_i} \vee^+ \cdots \vee^+ J_j'^1 \vee^+ \cdots J_j'^{l_j} \vee^+ \cdots) \vdash}\,\text{Release}}{\Gamma''' \Downarrow \overline{\forall x}.\ L'^{\perp} \vee^+ \cdots \vee^+ I_i'^1 \vee^+ \cdots I_i'^{k_i} \vee^+ \cdots \vee^+ J_j'^1 \vee^+ \cdots J_j'^{l_j} \vee^+ \cdots \vdash}\,\Downarrow\forall\vdash}{\Gamma''' \Uparrow \vdash}\,\text{Focus}$$

Figure 6. Derivations for Lemma 4.9