On proof-terms for (minimal) deduction modulo

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Motivations

- Which amount of information should we keep in proof-terms ?

  $\longrightarrow$ Church-style vs. Curry-style

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- Pure Type Systems vs. Type Assignment Systems

Syntax

	P_0	P_1	P_2	P_3
Syntax	α $\boldsymbol{\lambda\alpha{:}A.\pi}$ $\pi\pi'$ $\lambda x.\pi$ πt	α $\lambda\alpha.\pi$ $\pi\pi'$ $\boldsymbol{\lambda x.\pi}$ $\boldsymbol{\pi t}$	α $\lambda\alpha.\pi$ $\pi\pi'$ $\boldsymbol{I(\pi)}$ $\boldsymbol{E(\pi)}$	α $\lambda\alpha.\pi$ $\pi\pi'$
Formulas in proofs	✓	✗	✗	✗
Terms in proofs	✓	✓	✗	✗
$\forall$-cuts	✓	✓	✓	✗
$\Rightarrow$-cuts	✓	✓	✓	✓
Type-checking decidability	✓	✗	✗	✗

From Church to Curry

Typing

axiom	$\mathbf{P}_0, \mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3$	$\dfrac{}{\Gamma, \alpha : A \vdash \alpha : B}\ A \equiv B$
$\Rightarrow$ **-elim**	$\mathbf{P}_0, \mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3$	$\dfrac{\Gamma \vdash \pi : C \qquad \Gamma' \vdash \pi' : A}{\Gamma\Gamma' \vdash (\pi\ \pi') : B}\ C \equiv A \Rightarrow B$
$\Rightarrow$ **-intro**	$\mathbf{P}_0$	$\dfrac{\Gamma, \alpha : A \vdash \pi : B}{\Gamma \vdash \lambda\alpha{:}A.\ \pi : C}\ C \equiv A \Rightarrow B$
	$\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3$	$\dfrac{\Gamma, \alpha : A \vdash \pi : B}{\Gamma \vdash \lambda\alpha.\pi : C}\ C \equiv A \Rightarrow B$

$\forall$-elim	$\mathbf{P_0, P_1}$	$\dfrac{\Gamma \vdash \pi : B}{\Gamma \vdash \boldsymbol{\pi}\ \mathbf{t} : C}$	$B \equiv \forall x.A, \ \ C \equiv (t/x)A$
	$\mathbf{P_2}$	$\dfrac{\Gamma \vdash \pi : B}{\Gamma \vdash \mathbf{E}(\boldsymbol{\pi}) : C}$	$B \equiv \forall x.A, \ \ C \equiv (t/x)A$
	$\mathbf{P_3}$	$\dfrac{\Gamma \vdash \pi : B}{\Gamma \vdash \boldsymbol{\pi} : C}$	$B \equiv \forall x.A, \ \ C \equiv (t/x)A$

$\forall$-intro	$\mathbf{P_0, P_1}$	$\dfrac{\Gamma \vdash \pi : A}{\Gamma \vdash \boldsymbol{\lambda}\mathbf{x}.\boldsymbol{\pi} : B}$	$B \equiv \forall x.A, \ \ x \notin FV(\Gamma)$
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Erasing functions

$\mathbf{P_0 \longrightarrow P_1}$	$\mathbf{P_1 \longrightarrow P_2}$	$\mathbf{P_2 \longrightarrow P_3}$
$\lvert\alpha\rvert_0^1 = \alpha$	$\lvert\alpha\rvert_1^2 = \alpha$	$\lvert\alpha\rvert_2^3 = \alpha$
$\boldsymbol{\lvert\lambda\alpha : A.\pi\rvert_0^1 = \lambda\alpha.\lvert\pi\rvert_0^1}$	$\lvert\lambda\alpha.\pi\rvert_1^2 = \lambda\alpha.\lvert\pi\rvert_1^2$	$\lvert\lambda\alpha.\pi\rvert_2^3 = \lambda\alpha.\lvert\pi\rvert_2^3$
$\lvert\pi\pi'\rvert_0^1 = \lvert\pi\rvert_0^1\ \lvert\pi'\rvert_0^1$	$\lvert\pi\pi'\rvert_1^2 = \lvert\pi\rvert_1^2\ \lvert\pi'\rvert_1^2$	$\lvert\pi\pi'\rvert_2^3 = \lvert\pi\rvert_2^3\ \lvert\pi'\rvert_2^3$
$\lvert\lambda x.\pi\rvert_0^1 = \lambda x.\lvert\pi\rvert_0^1$	$\boldsymbol{\lvert\lambda x.\pi\rvert_1^2 = I(\lvert\pi\rvert_1^2)}$	$\boldsymbol{\lvert I(\pi)\rvert_2^3 = \pi}$
$\lvert\pi t\rvert_0^1 = \lvert\pi\rvert_0^1 t$	$\boldsymbol{\lvert\pi t\rvert_1^2 = E(\lvert\pi\rvert_1^2)}$	$\boldsymbol{\lvert E(\pi)\rvert_2^3 = \pi}$

For all $0 \leq i < j \leq 3$:

- If $\Gamma \vdash_i \pi : A$, then $\Gamma \vdash_j |\pi|_i^j : A$.

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- If $\Gamma \vdash_i \pi : A$, then $\Gamma \vdash_j |\pi|_i^j : A$.
- If $\Gamma \vdash_j \pi' : A$,
  then there exists π such that
  $$|\pi|_i^j = \pi' \text{ and } \Gamma \vdash_i \pi : A.$$

β-Reduction

$\mathbf{P_0}$	$(\lambda\alpha : A.\pi)\pi' \quad \to \quad (\pi'/\alpha)\pi$	$(\lambda x.\pi)t \quad \to \quad (t/x)\pi$
$\mathbf{P_1}$	$(\lambda\alpha.\pi)\pi' \quad \to \quad (\pi'/\alpha)\pi$	$(\lambda x.\pi)t \quad \to \quad (t/x)\pi$
$\mathbf{P_2}$	$(\lambda\alpha.\pi)\pi' \quad \to \quad (\pi'/\alpha)\pi$	$E(I(\pi)) \quad \to \quad \pi$
$\mathbf{P_3}$	$(\lambda\alpha.\pi)\pi' \quad \to \quad (\pi'/\alpha)\pi$	

$\longrightarrow$ β-reduction does not model $\forall$-cuts in system $\mathbf{P_3}$

SN in $\mathbf{P_0, P_1, P_2}$

$$|SN_0|_0^1 = SN_1 \qquad |SN_1|_1^2 = SN_2$$

A theory $\mathcal{T}$ is strongly normalizing in system P_0
iff it is strongly normalizing in system P_1
iff it is strongly normalizing in system P_2

(from equivalence of provability)

SN in P_2, P_3

$$|SN_2|_2^3 \supsetneq SN_3$$

$$SN_2 \ni (E(\lambda\alpha.\alpha\alpha))\,(\lambda\alpha.\alpha\alpha) \longrightarrow_{|\cdot|_2^3} (\lambda\alpha.\alpha\alpha)\,(\lambda\alpha.\alpha\alpha) \notin SN_3$$

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What about theories?

(ill-typed couter-example)

$\mathbf{P_3}$ SN $\Rightarrow$ $\mathbf{P_2}$ SN

Given a theory,

if it is strongly normalizing in system $\mathbf{P_3}$,

let $(\pi_i)_{i \in \mathbf{N}}$ a reductions sequence in $\mathbf{P_2}$ with $\Gamma \vdash_2 \pi_0 : A$

then $(|\pi_i|_2^3)_{i \in \mathbf{N}}$ is finite.

Since $\Rightarrow$-reductions are translated to $\Rightarrow$-reductions,

$(\pi_i)_{i \in \mathbf{N}}$ contains an infinite subesequence of $\forall$-reductions.

That's absurd.

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$\mathbf{P_3}$ pre-model $\Rightarrow$ theory SN in all systems

P_3 (complete) pre-model

- **Church** $\qquad \dfrac{\Gamma \vdash \pi : B}{\Gamma \vdash \pi\ t : C} \quad B \equiv \forall x.A,\ \ C \equiv (t/x)A$

$$[\![\forall x.A]\!]_\varphi = \{\pi,\ \ \forall t,\ \pi t \in [\![A]\!]_{\varphi+\langle x,t\rangle}\} \ \triangleq\ \tilde{\forall}\{[\![A]\!]_{\varphi+\langle x,t\rangle},\ t\}$$

with $\tilde{\forall}\mathcal{E} \ \triangleq\ \{\pi$ such that $\forall t\ \forall E \in \mathcal{E},\ \pi t \in E(t)\}$

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with $\tilde{\forall}\mathcal{E} \ \triangleq\ \bigcap \mathcal{E}$

Church SN $\Rightarrow$ Curry SN ?

Curry $\underline{\textbf{computes more}}$ **than Church**

$\longrightarrow$ even for well-typed terms :

$$(I(\lambda\alpha.\alpha))\ \beta \quad \longrightarrow_{|\cdot|_2^3} \quad (\lambda\alpha.\alpha)\ \beta$$

with $\quad \beta : A \vdash_2 (I(\lambda\alpha.\alpha))\ \beta : \forall x.A \quad$ **if** $\quad \forall x.(A \Rightarrow A) \equiv A \Rightarrow \forall x.A$

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with $\beta : A \vdash_2 (I(\lambda\alpha.\alpha))\ \beta : \forall x.A$ **if** $\forall x.(A \Rightarrow A) \equiv A \Rightarrow \forall x.A$

$$\forall x.(A \Rightarrow B_x) \;\equiv\; A \Rightarrow \forall x.B_x$$

always true in Curry-style

false *à priori* in Church-style

An issue

$\longrightarrow$ *restrain* Curry-style not to compute more than Church-style

$\longrightarrow$ **non confusing** theories

$$\forall x.A \;\not\equiv\; B \Rightarrow C$$

$\longrightarrow$ avoid "stopping computation" well-typed terms in P_2 :

$$I(\pi) \qquad E(\pi) \qquad (\lambda\alpha.\pi)\ t$$

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If a non-confusing theory is SN in P_2 then it is SN in P_3.

Outline

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  always implies Church-style strong normalization
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- Computation is equivalent for non-confusing theories

- Computation is not strong normalisation
  (i.e. SN can be equivalent even if computation is not)
  ($\rightarrow$ can we get rid of the non-confusion hypothesis ?)