Generalizing Boolean Algebras for Deduction Modulo

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Introduction

- extend the notion of **superconsistency**
  - consistency:

    A theory $\mathfrak{A}$ is **consistent** if there exists **a** model $\mathcal{M}$ in which there exists **an** interpretation $[\![\_]\!]$ where:

$$[\![\mathfrak{A}]\!] \neq \bot$$

  - super-consistency:

    A theory $\mathfrak{A}$ is **super-consistent** if for **all** model $\mathcal{M}$, there exists **an** interpretation $[\![\_]\!]$ where:

$$[\![\mathfrak{A}]\!] \cong \top$$

Superconsistency

- what is a **theory** ?
  - rewriting systems of Deduction Modulo
  - a congruence on propositions generated by a set of rewrite rules

$$
\begin{array}{rcl}
x + 0 & \longrightarrow & 0 \\
P(0) & \longrightarrow & \forall x P(x)
\end{array}
$$

- what is a **model** ?
  - **intuitionistic** setting: **Heyting** algebras
  - need to **generalize** over it: **pre**-Heyting algebras (plus technical conditions)
  - pre-Heyting algebras are still sound and complete
- what do we **get** ?
  - reducibility candidates **are** a pre-Heyting algebra (and not a Heyting algebra)
  - all super-consistent theories have the **normalization** property

- **extend** the notion of super-consistency
  - to classical logic
  - to sequent calculus
  - to proofs of cut admissibility
- of course, super-consistency implies cut-admissibility in classical sequent calculus modulo.
  - but through a $\neg\neg$-translation **and** a back and forth translation in Natural Deduction [Dowek-Werner]
  - direct proof wanted

Introduction

- the framework:
  - monolateral classical sequent calculus
  - deduction modulo with explicit conversion rule
  - negation is an operation and not a connector:

$$(A \wedge B)^{\perp} = A^{\perp} \wedge B^{\perp}$$

- the method: sequent reducibility candidates [Dowek, Hermant].

Pre-Boolean algebras

- weaken the order of a Boolean Algebra into a pre-order ($a \leq b$ and $b \leq a$)
- keep the same axioms

$$a \leq a \vee b \qquad\qquad b \leq a \vee b$$
$$a \leq c \text{ and } b \leq c \quad \text{implies} \quad a \vee b \leq c$$

- more strict than [Dowek]: $a^{\perp\perp} = a$ (and not $a^{\perp\perp} \cong a$)
- in fact, even no need for the pre-order $\leq$:
  - we always consider a trivial pre-order ($a \leq b$ for any a, b)
  - and no need for any Boolean Algebra axiom ...
- **classical super-consistency**: to have a model interpretation $[\![\_]\!]$ in **any** pre-Boolean algebra.
  - only condition on $[\![\_]\!]$:

$$A \equiv B \quad \text{implies} \quad [\![A]\!] = [\![B]\!]$$

The Plan

- find a nice pre-Boolean algebra
- interpret sequents in the pre-Boolean algebra
- prove adequacy lemma
- of course, no (strong) normalization

Inheritage from Linear Logic [Okada, Brunel]

- identifying a **site** (stoup) in sequents: pointed sequents

$$\vdash \Delta, A^\circ$$

- interaction $\star$:

$$\begin{aligned} \vdash \Delta_1, A^\circ \;\star\; \vdash \Delta_2, A^{\perp\circ} &= \vdash \Delta_1, \Delta_2 \\ \vdash \Delta_1, A^\circ \;\star\; X &= \{ \vdash \Delta_1, \Delta_2 \mid \vdash \Delta_2, A^{\perp\circ} \in X \} \end{aligned}$$

- define an object having **what we want**: $\perp\!\!\!\perp$ (cut-free provable sequents)
- define an **orthogonality** operation for a set of sequents:

$$X^\perp = \{ \vdash \Delta, A^\circ \mid \vdash \Delta, A^\circ \;\star\; X \subseteq \perp\!\!\!\perp \}$$

  - usual properties of an orthogonality operation:

$$\begin{aligned} X &\subseteq X^{\perp\perp} \\ X \subseteq Y &\Rightarrow Y^\perp \subseteq X^\perp \\ X^{\perp\perp\perp} &= X^\perp \end{aligned}$$

Inheritage from Linear Logic [Okada, Brunel]

- the **domain** of interpretation D:

$$Ax^\circ \subseteq X \subseteq \perp\!\!\!\perp$$

  - X has to be **stable** ($i.e$ $X^{\perp\perp} = X$)
  - **CR3** (neutral proof terms): $Ax^\circ \subseteq X$
  - **CR1** (SN proof terms): $X \subseteq \perp\!\!\!\perp$
  - no **CR2** (sequents)
- core operation + orthogonality:

$$\begin{aligned} X.Y &= \{ \vdash \Delta_A, \Delta_B, (A \wedge B)^\circ \mid (\vdash \Delta_A, A^\circ) \in X \\ &\qquad\qquad \text{and } (\vdash \Delta_B, B^\circ) \in Y \} \\ X \wedge Y &= \{X.Y \cup Ax^\circ\}^{\perp\perp} \end{aligned}$$

it is pre-Boolean algebra

- nothing to check on $\leq$ (we dropped it !)
- stability of D under $(.)^{\perp}$, $\wedge$, ...
- stability of elements of D under $\equiv$

Super-consistency and Adequacy

Super-consistency:

- give us an interpretation such that $A \equiv B$ implies $[\![A]\!] = [\![B]\!]$

Adequacy:

- takes a proof of $\vdash A_1, ..., A_n$
- assumes $\vdash \Delta_i, (A_i^\perp)^\circ \in A_i^{*\perp}$
- ensures $\vdash \Delta_1, ..., \Delta_n \in \perp\!\!\!\perp$

Features of adequacy:

- conversion rule: processed by the SC condition
- axiom rule: we must have $\vdash A^\perp, A^\circ \in A^* \Rightarrow$ untyped candidates because of super-consistency.

Directly implies **cut-elimination**.

Extracting a Boolean algebra

$A_1, ..., A_n \in \lfloor A \rfloor$ iff

- assume $\vdash \Delta_i, (A_i^\perp)^\circ \in A_i^{*\perp}$
- then $\vdash \Delta_1, ..., \Delta_n, A^\circ \in A^*$
- equivalently, for any $\vdash \Delta, A^{\perp\circ} \in A^{*\perp}$, $\vdash \Delta_1, ..., \Delta_n, \Delta \in \perp\!\!\!\perp$

Operations:

- $\lfloor A \rfloor \wedge \lfloor B \rfloor = \lfloor A \wedge B \rfloor$
- $\forall \{\lfloor A[t/x] \rfloor\} = \lfloor \forall x A \rfloor$
- ...

This is a Boolean Algebra (not complete !)