A decomposition approach for the discrete-time approximation of BSDEs with a jump II: the quadratic case

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Abstract

We study a discrete-time approximation for solutions of forward-backward stochastic differential equations (FBSDEs) with a jump. Assuming that the generators are Lipschitz or with a quadratic growth w.r.t. the variable z, and that the terminal conditions are bounded, we prove the convergence of the scheme when the number of time steps n goes to infinity. We present a method based on the companion paper [14] which allows to link the FBSDE with a jump with a system of recursive Brownian FBSDEs, then, we use the classical result on the Brownian FBSDEs to approximate the system of recursive FBSDEs. That allows to get a convergence rate similar to the convergence rate for the schemes of Brownian FBSDEs.

Keywords: discrete-time approximation, forward-backward stochastic differential equation with a jump, Lipschitz generator, generator of quadratic growth, progressive enlargement of filtrations, decomposition in the reference filtration.

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1 Introduction

In this paper, we study a discrete-time approximation scheme for the solution of a system of forward-backward stochastic differential equations (FBSDEs) with a jump of the form

$$\begin{cases} X_t &= x + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s + \int_0^t \beta(s, X_{s^-})dH_s \;, \\ Y_t &= g(X_T) + \int_t^T f(s, X_s, Y_s, Z_s, U_s)ds - \int_t^T Z_s dW_s - \int_t^T U_s dH_s \;, \end{cases}$$

where $H_t = \mathbb{1}_{\tau \leq t}$ and τ is a jump time, which can represent a default time in credit risk or counterparty risk. Such equations naturally appear in finance, see for example Bielecki and Jeanblanc [2], Lim and Quenez [18], Ankirchner *et al.* [1] for an application to exponential utility maximization and Kharroubi and Lim [14] for the hedging problem in a complete market. We study the cases where the generator f is Lipschitz and f has a quadratic growth w.r.t. z.

To the best of our knowledge, there is none work which studies the discrete-time approximation scheme of these FBSDEs. For the FBSDEs with jumps, there is only the paper of Bouchard and Elie [4], but in this paper the jump is a Poisson measure independent of the Brownian motion. Notice that the problem of discretization of the FBSDEs without jump with a Lipschitz generator is well understood, see e.g. Chevance [5], Bouchard and Touzi [3], Zhang [23], Gobet *et al.* [8], Delarue and Menozzi [6]. For the case with a generator of quadratic growth w.r.t. z there are very few works on. As far as we know, the only works where the time approximation is studied are the papers of Imkeller *et al.* [9] and Richou [22].

In this paper, we can not use the same technics as [4] because they use the Malliavin calculus to get some regularity results and in our framework, to the best of our knowledge, there is none work about the Malliavin calculus. To get a discrete-time approximation scheme we use the results of [14], that allows to decompose the FBSDE with a jump in a recursive system of Brownian FBSDEs. Then, we give a discrete-time approximation scheme for the solutions of each Brownian FBSDE, for that we use the classical results if the generator is Lipschitz and the results of Richou if the generator has a quadratic growth w.r.t. z. Finally, we obtain a discrete-time approximation scheme by combining the schemes for the Brownian FBSDEs.

The paper is organized as follows. The next section presents the FBSDE, the different assumptions on the coefficients and the functions appearing in the FBSDE and we recall the result of [14]. In Section 3, we give a discrete-time approximation scheme for the FBSDEs with a Lipschitz generator and we obtain a global error estimate. In Section 4, we give a discrete-time approximation scheme for the FBSDEs with a generator of quadratic growth w.r.t. z and we obtain a global error estimate.

2 Preliminaries

2.1 Notation

Throughout this paper, we let $(\Omega, \mathcal{G}, \mathbb{P})$ a probability space on which is defined a standard one dimensional Brownian motion W. We denote $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ the natural filtration of W, augmented by all the $\mathbb{P}$-null sets. We also consider on this space a random time τ, i.e. a nonnegative $\mathcal{G}$-measurable random variable, and we denote classically the associated jump process by H which is given by

$$H_t \;=\; \mathbb{1}_{\tau \leq t}\;, \qquad t \geq 0\;.$$

We denote by $\mathbb{D} = (\mathcal{D}_t)_{t\geq 0}$ the smallest right-continuous filtration for which τ is a stopping time. The global information is then defined by the progressive enlargement $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ of the initial filtration where $\mathbb{G} := \mathbb{F} \vee \mathbb{D}$. This kind of enlargement was studied by Jacod, Jeulin and Yor in the 80s (see e.g. [11], [12] and [10]). We introduce some notations used throughout the paper:

- $\mathcal{P}(\mathbb{F})$ (resp. $\mathcal{P}(\mathbb{G})$) is the σ-algebra of $\mathbb{F}$ (resp. $\mathbb{G}$)-predictable measurable subsets of $\Omega \times \mathbb{R}_+$, i.e. the σ-algebra generated by the left-continuous $\mathbb{F}$ (resp. $\mathbb{G}$)-adapted processes,
- $\mathcal{PM}(\mathbb{F})$ (resp. $\mathcal{PM}(\mathbb{G})$) is the σ-algebra of $\mathbb{F}$ (resp. $\mathbb{G}$)-progressively measurable subsets of $\Omega \times \mathbb{R}_+$.

We shall make, throughout the sequel, the standing assumption in the progressive enlargement of filtrations known as density assumption (see e.g. [13, **?**, 14]).

(DH) There exists a positive and bounded $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+)$-measurable process γ such that

$$\mathbb{P}\big[\tau \in d\theta \,\big|\, \mathcal{F}_t\big] \;=\; \gamma_t(\theta)d\theta\;, \qquad t \geq 0\;.$$

Using Proposition 2.1 in [14] we get that **(DH)** ensures that the process H admits an intensity.

Proposition 2.1. *The process H admits a compensator of the form $\lambda_t dt$, where the process λ is defined by*

$$\lambda_t \;=\; \frac{\gamma_t(t)}{\mathbb{P}\big[\tau > t \,\big|\, \mathcal{F}_t\big]}\mathbb{1}_{t \leq \tau}\;, \qquad t \geq 0\;.$$

We impose the following assumption to the process λ:

(HBI) The process λ is bounded.

We also introduce the martingale invariance assumption known as the **(H)**-hypothesis.

(H) Any $\mathbb{F}$-martingale remains a $\mathbb{G}$-martingale.

We now introduce the following spaces, where $a, b \in \mathbb{R}_+$ with $a \leq b$, and $T < \infty$ is the terminal time:

– $\mathcal{S}^\infty_{\mathbb{G}}[a,b]$ (resp. $\mathcal{S}^\infty_{\mathbb{F}}[a,b]$) is the set of $\mathcal{PM}(\mathbb{G})$ (resp. $\mathcal{PM}(\mathbb{F})$)-measurable processes $(Y_t)_{t\in[a,b]}$ essentially bounded:

$$\|Y\|_{\mathcal{S}^\infty[a,b]} \;:=\; \operatorname*{ess\,sup}_{t\in[a,b]}|Y_t| \;<\; \infty\,.$$

– $\mathcal{S}^p_{\mathbb{G}}[a,b]$ (resp. $\mathcal{S}^p_{\mathbb{F}}[a,b]$), with $p\geq 2$, is the set of $\mathcal{PM}(\mathbb{G})$ (resp. $\mathcal{PM}(\mathbb{F})$)-measurable processes $(Y_t)_{t\in[a,b]}$ such that

$$\|Y\|_{\mathcal{S}^p[a,b]} \;:=\; \Big(\mathbb{E}\Big[\sup_{t\in[a,b]}|Y_t|^p\Big]\Big)^{\frac{1}{p}} \;<\; \infty\,.$$

– $H^p_{\mathbb{G}}[a,b]$ (resp. $H^p_{\mathbb{F}}[a,b]$), with $p\geq 2$, is the set of $\mathcal{P}(\mathbb{G})$ (resp. $\mathcal{P}(\mathbb{F})$)-measurable processes $(Z_t)_{t\in[a,b]}$ such that

$$\|Z\|_{H^p[a,b]} \;:=\; \mathbb{E}\Big[\Big(\int_a^b|Z_t|^2dt\Big)^{\frac{p}{2}}\Big]^{\frac{1}{p}} \;<\; \infty\,.$$

– $L^2(\lambda)$ is the set of $\mathcal{P}(\mathbb{G})$-measurable processes $(U_t)_{t\in[0,T]}$ such that

$$\|U\|_{L^2(\mu)} \;:=\; \Big(\mathbb{E}\Big[\int_0^T|U_s|^2\lambda_sds\Big]\Big)^{\frac{1}{2}} \;<\; \infty\,.$$

2.2 Forward-Backward SDE with a jump

Given measurable functions $b:[0,T]\times\mathbb{R}\to\mathbb{R}$, $\sigma:[0,T]\to\mathbb{R}$, $\beta:[0,T]\times\mathbb{R}\to\mathbb{R}$, $g:\mathbb{R}\to\mathbb{R}$ and $f:[0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, and an initial condition $x\in\mathbb{R}$, we study the discrete-time approximation of the solution (X,Y,Z,U) in $\mathcal{S}^2_{\mathbb{G}}[0,T]\times\mathcal{S}^\infty_{\mathbb{G}}[0,T]\times H^2_{\mathbb{G}}[0,T]\times L^2(\lambda)$ to the following forward-backward stochastic differential equation:

$$X_t \;=\; x+\int_0^t b(s,X_s)ds+\int_0^t\sigma(s)dW_s+\int_0^t\beta(s,X_{s^-})dH_s\,,\quad 0\leq t\leq T\,,\tag{2.1}$$

$$\begin{aligned}Y_t \;=\; & g(X_T)+\int_t^T f\big(s,X_s,Y_s,Z_s,U_s(1-H_s)\big)ds\\ &\qquad -\int_t^T Z_sdW_s-\int_t^T U_sdH_s\,,\quad 0\leq t\leq T\,,\end{aligned}\tag{2.2}$$

when the generator of BSDE (2.2) has a quadratic growth w.r.t. z.

Remark 2.1. In BSDE (2.2), the jump component U of the unknown (Y,Z,U) appears in the generator f with an additional multiplicative term $1-H$. This ensures the equation to be well posed in $\mathcal{S}^\infty_{\mathbb{G}}[0,T]\times H^2_{\mathbb{G}}[0,T]\times L^2(\lambda)$. Indeed, the component U lives in $L^2(\lambda)$, thus its value on $(\tau\wedge T,T]$ is not defined, since the intensity λ vanishes on $(\tau\wedge T,T]$. We therefore introduce the term $1-H$ to kill the value of U on $(\tau\wedge T,T]$ and hence to avoid making the equation depends on it.

We first prove that the decoupled system (2.1)-(2.2) admits a solution. To this end, we introduce several assumptions on the coefficients b, σ, β, g and f. We consider the following assumptions for the forward coefficients:

(HF) There exist two constants K_a and L_a such that the functions b, σ and β satisfy

$$|b(t,0)| + |\sigma(t)| + |\beta(t,0)| \;\leq\; K_a \;,$$

and

$$|b(t,x) - b(t,x')| + |\beta(t,x) - \beta(t,x')| \;\leq\; L_a|x - x'| \;,$$

for all $(t,x,x') \in [0,T] \times \mathbb{R} \times \mathbb{R}$.

For the backward coefficients g and f, we consider the following assumptions:

(HBQ) There exist two constants M_g and K_q such that the functions g and f satisfy

$$|g(x)| \;\leq\; M_g \;,$$

and

$$|f(t,x,y,z,u)| \;\leq\; K_q\big(1 + |y| + |z|^2 + |u|\big) \;,$$

for all $(t,x,y,z,u) \in [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, and the function $f(t,x,y,.,u)$ is convexe or concave uniformly in $(t,x,y,u) \in [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$.

Following the decomposition approach of [14], we introduce the recursive system of FBSDEs associated with (2.1)-(2.2):

- Find $(X^1(\theta), Y^1(\theta), Z^1(\theta)) \in \mathcal{S}^2_{\mathbb{F}}[0,T] \times \mathcal{S}^\infty_{\mathbb{F}}[\theta,T] \times H^2_{\mathbb{F}}[\theta,T]$ such that

$$X^1_t(\theta) \;=\; x + \int_0^t b\big(s, X^1_s(\theta)\big)ds + \int_0^t \sigma(s)dW_s + \beta\big(\theta, X^1_{\theta^-}(\theta)\big)\mathbb{1}_{\theta\leq t} \;, \quad 0 \leq t \leq T \;, \tag{2.3}$$

$$Y^1_t(\theta) \;=\; g\big(X^1_T(\theta)\big) + \int_t^T f\big(s, X^1_s(\theta), Y^1_s(\theta), Z^1_s(\theta), 0\big)ds - \int_t^T Z^1_s(\theta)dW_s \;, \quad \theta \leq t \leq T \;, \tag{2.4}$$

for all $\theta \in [0,T]$.

- Find $(X^0, Y^0, Z^0) \in \mathcal{S}^2_{\mathbb{F}}[0,T] \times \mathcal{S}^\infty_{\mathbb{F}}[0,T] \times H^2_{\mathbb{F}}[0,T]$ such that

$$X^0_t \;=\; x + \int_0^t b\big(s, X^0_s\big)ds + \int_0^t \sigma(s)dW_s \;, \quad 0 \leq t \leq T \;, \tag{2.5}$$

$$Y^0_t \;=\; g\big(X^0_T\big) + \int_t^T f\big(s, X^0_s, Y^0_s, Z^0_s, Y^1_s(s) - Y^0_s\big)ds - \int_t^T Z^0_s dW_s \;, \quad 0 \leq t \leq T \;. \tag{2.6}$$

Then, the link between FBSDE (2.1)-(2.2) and the recursive system of FBSDEs (2.5)-(2.6) and (2.3)-(2.4) is given by the following result.

Theorem 2.1. *Assume that **(HF)** and **(HBQ)** hold true. Then, FBSDE* (2.1)-(2.2) *admits a unique solution* $(X,Y,Z,U) \in \mathcal{S}^2_{\mathbb{G}}[0,T] \times \mathcal{S}^\infty_{\mathbb{G}}[0,T] \times H^2_{\mathbb{G}}[0,T] \times L^2(\lambda)$ *given by*

$$\left\{\begin{array}{rcl} X_t &=& X^0_t \mathbb{1}_{t<\tau} + X^1_t(\tau)\mathbb{1}_{\tau\leq t} \;, \\ Y_t &=& Y^0_t \mathbb{1}_{t<\tau} + Y^1_t(\tau)\mathbb{1}_{\tau\leq t} \;, \\ Z_t &=& Z^0_t \mathbb{1}_{t\leq\tau} + Z^1_t(\tau)\mathbb{1}_{\tau< t} \;, \\ U_t &=& \big(Y^1_t(t) - Y^0_t\big)\mathbb{1}_{t\leq\tau} \;, \end{array}\right. \tag{2.7}$$

where $(X^1(\theta), Y^1(\theta), Z^1(\theta))$ *is the unique solution to FBSDE* (2.3)-(2.4) *in* $\mathcal{S}^2_{\mathbb{F}}[0,T]\times\mathcal{S}^\infty_{\mathbb{F}}[\theta,T]\times H^2_{\mathbb{F}}[\theta,T]$*, for* $\theta \in [0,T]$*, and* (X^0,Y^0,Z^0) *is the unique solution to FBSDE* (2.5)-(2.6) *in* $\mathcal{S}^2_{\mathbb{F}}[0,T] \times \mathcal{S}^\infty_{\mathbb{F}}[0,T] \times H^2_{\mathbb{F}}[0,T]$.

Proof. The existence and uniqueness of the forward process X and its link with X^0 and X^1 have already been proved in the first part of this work [15]. We now concentrate on the backward equation.
To follow the decomposition approach initiated by the authors in [14], we need the generator to be predictable. To this end, we notice that in BSDE (2.2), we can replace the generator $(t,y,z,u) \mapsto f(t,X_t,y,z,u(1-H_t))$ by the predictable map $(t,y,z,u) \mapsto f(t,X_{t^-},y,z,u(1-H_{t^-}))$.
Suppose that **(HBQ)** holds true. The existence of a solution $(Y,Z,U) \in \mathcal{S}^2_{\mathbb{G}}[0,T]\times H^2_{\mathbb{G}}[0,T]\times L^2(\lambda)$ is then a direct consequence of Proposition 3.1 in [14]. We then notice that from the definition of H we have $f(t,x,y,z,u(1-H_t)) = f(t,x,y,z,0)$ for all $t \in (\tau \wedge T, T]$. This property and **(HBQ)** allow to apply Theorem 4.2 in [14], which gives the uniqueness of a solution to BSDE (2.2). □

Throughout the sequel, we give an approximation of the solution to FBSDE (2.1)-(2.2) by studying the approximation of the solutions to the recursive system of FBSDEs (2.3)-(2.4) and (2.5)-(2.6). For that we use the recent results of [22] about the discretization of BSDEs in the case where the driver is quadratic.

3 Discrete-time scheme for the FBSDE

In this section, we introduce a discrete-time approximation of the solution (X,Y,Z,U) to FBSDE (2.1)-(2.2) based on its decomposition given by Theorem 2.1. For that we set $\epsilon \in (0,T)$ and $N \in \mathbb{N}$. We use a non equidistant grid π with $2n+1$ discretization times. The $n+1$ first discretization times are defined on $[0,T-\epsilon]$ by

$$t_k \;:=\; T\Big(1-\big(\frac{\epsilon}{T}\big)^{k/n}\Big), \quad 0 \leq k \leq n \;,$$

and we use an equidistant net on $[T-\epsilon,T]$ for the last n discretization times

$$t_k \;:=\; T - \Big(\frac{2n-k}{n}\Big)\epsilon \;, \quad n < k \leq 2n \;.$$

For $t \in [0,T]$, we denote by $\pi(t)$ (resp. $\pi_+(t)$) the largest (resp. smallest) element of π smaller (resp. larger) than t:

$$\begin{aligned}
\pi(t) &:= \max\left\{ t_i \ , \ i = 0, \ldots, 2n \mid t_i \leq t \right\} \\
(\text{resp. } \pi_+(t) &:= \min\left\{ t_i \ , \ i = 0, \ldots, 2n \mid t_i \geq t \right\}) \ .
\end{aligned}$$

We also denote by $|\pi|$ the mesh of π:

$$|\pi| \ := \ \max\left\{ t_{i+1} - t_i \ , \ i = 0, \ldots, 2n-1 \right\} ,$$

and by ΔW_i^π (resp. Δt_i^π) the increment of W (resp. the difference) between t_i and t_{i-1}: $\Delta W_i^\pi := W_{t_i} - W_{t_{i-1}}$ (resp. $\Delta t_i^\pi := t_i - t_{i-1}$), for $1 \leq i \leq 2n$.

3.1 Discrete-time scheme for X

We introduce an approximation of X based on the discretization of the processes X^0 and X^1.

• *Euler scheme for* X^0. We consider the classical scheme $X^{0,\pi}$ defined by

$$\begin{cases} X_{t_0}^{0,\pi} \ = \ x \ , \\ X_{t_i}^{0,\pi} \ = \ X_{t_{i-1}}^{0,\pi} + b(t_{i-1}, X_{t_{i-1}}^{0,\pi})\Delta t_i^\pi + \sigma(t_{i-1})\Delta W_i^\pi \ , \quad 1 \leq i \leq 2n \ . \end{cases} \tag{3.8}$$

• *Euler scheme for* X^1. Since the process X^1 depends on two parameters t and θ, we introduce a discretization of X^1 in these two variables. We then consider the following scheme

$$\begin{cases} X_{t_0}^{1,\pi}(\pi(\theta)) \ = \ x + \beta(t_0, x)\mathbb{1}_{\pi(\theta)=0} \ , \quad 0 \leq k \leq 2n \ , \\ X_{t_i}^{1,\pi}(\pi(\theta)) \ = \ X_{t_{i-1}}^{1,\pi}(\pi(\theta)) + b(t_{i-1}, X_{t_{i-1}}^{1,\pi}(\pi(\theta)))\Delta t_i^\pi + \sigma(t_{i-1})\Delta W_i^\pi \\ \qquad\qquad + \beta(t_{i-1}, X_{t_{i-1}}^{1,\pi}(\pi(\theta)))\mathbb{1}_{t_i=\pi(\theta)} \ , \qquad 1 \leq i \leq 2n \ , \quad 0 \leq \theta \leq T \ . \end{cases} \tag{3.9}$$

We are now able to provide an approximation of the process X solution to FSDE (2.1). We consider the scheme X^π defined by

$$X_t^\pi \ = \ X_{\pi(t)}^{0,\pi}\mathbb{1}_{t<\tau} + X_{\pi(t)}^{1,\pi}(\pi(\tau))\mathbb{1}_{t\geq\tau} \ , \quad 0 \leq t \leq T \ . \tag{3.10}$$

We shall denote by $\{\mathcal{F}_i^{0,\pi}\}_{0\leq i\leq 2n}$ (resp. $\{\mathcal{F}_i^{1,\pi}(\theta)\}_{0\leq i\leq 2n}$) the discrete-time filtration associated with $X^{0,\pi}$ (resp. $X^{1,\pi}$)

$$\begin{aligned}
\mathcal{F}_i^{0,\pi} &:= \sigma(X_{t_j}^{0,\pi}, \ j \leq i) \\
(\text{resp. } \mathcal{F}_i^{1,\pi}(\theta) &:= \sigma(X_{t_j}^{1,\pi}(\theta), \ j \leq i)) \ .
\end{aligned}$$

3.2 Discrete-time scheme for (Y, Z, U)

We introduce an approximation of (Y, Z) based on the discretization of (Y^0, Z^0) and (Y^1, Z^1). To this end we introduce the backward implicit schemes on π associated with BSDEs (2.4) and (2.6). Since the system is recursively coupled, we first introduce the scheme associated with (2.4). We then use it to define the scheme associated with (2.6). Before to give the schemes we define $\rho_s : \mathbb{R} \to \mathbb{R}$ the projection ball

$$B\Big(0, M_{z,1} + \frac{M_{z,2}}{(T-s)^{\frac{1}{2}}}\Big)$$

where $M_{z,1}$ and $M_{z,2}$ are some constants defined throughout the sequel. We also introduce g_N a Lipschitz approximation of g with Lipschitz constant N.

• *Backward scheme for* (Y^1, Z^1). We consider the implicit scheme $(Y^{1,\pi}, Z^{1,\pi})$ defined by

$$\begin{cases} Y_T^{1,\pi}(\pi(\theta)) &= g_N(X_T^{1,\pi}(\pi(\theta)))\ , \\ Z_{t_i}^{1,\pi}(\pi(\theta)) &= \rho_{t_{i+1}}\Big(\frac{1}{\Delta t_{i+1}^\pi}\mathbb{E}_i^{1,\pi(\theta)}\big[Y_{t_{i+1}}^{1,\pi}(\pi(\theta))\Delta W_{i+1}^\pi\big]\Big)\ , \qquad t_i \geq \pi(\theta)\ , \\ Y_{t_i}^{1,\pi}(\pi(\theta)) &= \mathbb{E}_i^{1,\pi(\theta)}\big[Y_{t_{i+1}}^{1,\pi}(\pi(\theta))\big] + f^\epsilon(t_i, X_{t_i}^{1,\pi}(\pi(\theta)), Y_{t_{i+1}}^{1,\pi}(\pi(\theta)), Z_{t_i}^{1,\pi}(\pi(\theta)))\Delta t_{i+1}^\pi\ , \end{cases} \tag{3.11}$$

where $\mathbb{E}_i^{1,\pi(\theta)} = \mathbb{E}[\ .\ |\mathcal{F}_i^{1,\pi}(\theta)]$ for $0 \leq i \leq 2n$ and $\theta \in [0, T]$, and

$$f^\epsilon(s, x, y, z) \ = \ \mathbb{1}_{s \leq T-\epsilon} f(s, x, y, z, 0) + \mathbb{1}_{s > T-\epsilon} f(s, x, y, 0, 0)\ .$$

• *Backward scheme for* (Y^0, Z^0). Since the generator of (2.6) involves the process $(Y_t^1(t))_{t\in[0,T]}$, we consider a discretization based on $Y^{1,\pi}$. We therefore consider the scheme $(Y^{0,\pi}, Z^{0,\pi})$ defined by

$$\begin{cases} Y_T^{0,\pi} &= g_N(X_T^{0,\pi})\ , \\ Z_{t_i}^{0,\pi} &= \rho_{t_{i+1}}\Big(\frac{1}{\Delta t_{i+1}^\pi}\mathbb{E}_i^0\big[Y_{t_{i+1}}^{0,\pi}\Delta W_{i+1}^\pi\big]\Big)\ , \qquad 0 \leq i \leq 2n-1\ , \\ Y_{t_i}^{0,\pi} &= \mathbb{E}_i^0\big[Y_{t_{i+1}}^{0,\pi}\big] + f^{\pi,\epsilon}(t_i, X_{t_i}^{0,\pi}, Y_{t_i}^{0,\pi}, Z_{t_i}^{0,\pi})\Delta t_{i+1}^\pi\ , \end{cases} \tag{3.12}$$

where $\mathbb{E}_i^0 = \mathbb{E}[\ .\ |\mathcal{F}_i^{0,\pi}]$ for $0 \leq i \leq n$, and $f^{\pi,\epsilon}$ is defined by

$$f^{\pi,\epsilon}(t, x, y, z) \ = \ \mathbb{1}_{t \leq T-\epsilon} f\big(t, x, y, z, Y_{\pi(t)}^{1,\pi}(\pi(t)) - y\big) + \mathbb{1}_{t > T-\epsilon} f\big(t, x, y, 0, Y_{\pi(t)}^{1,\pi}(\pi(t)) - y\big)\ ,$$

for all $(t, x, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$.
We then consider the following scheme for the solution (Y, Z, U) to BSDE (2.2)

$$\begin{cases} Y_t^\pi &= Y_{\pi(t)}^{0,\pi}\mathbb{1}_{t<\tau} + Y_{\pi(t)}^{1,\pi}(\pi(\tau))\mathbb{1}_{t\geq\tau}\ , \\ Z_t^\pi &= Z_{\pi(t)}^{0,\pi}\mathbb{1}_{t\leq\tau} + Z_{\pi(t)}^{1,\pi}(\pi(\tau))\mathbb{1}_{t>\tau}\ , \\ U_t^\pi &= \big(Y_{\pi(t)}^{1,\pi}(\pi(t)) - Y_{\pi(t)}^{0,\pi}\big)\mathbb{1}_{t\leq\tau}\ , \end{cases} \tag{3.13}$$

for $t \in [0, T]$.

4 Convergence of the scheme for the FSDE

We introduce the following assumption, which will be used to discretize X.

(HFD) There exist three constants K_b, K_t and K_β such that the functions b, σ and β satisfy

$$\begin{aligned}
\big|b(t,x)\big| &\leq K_b(1+|x|)\;, \\
\big|b(t,x)-b(t',x)\big| + \big|\sigma(t)-\sigma(t')\big| &\leq K_t\big(|t-t'|^{\frac{1}{2}} \wedge |t-t'|\big)\;, \\
\big|\beta(t,x)-\beta(t',x)\big| &\leq K_\beta|t-t'|\;,
\end{aligned}$$

for all $(t,t',x)\in[0,T]\times[0,T]\times\mathbb{R}$.

We now recall the error estimate for the scheme of X obtained in the first part of this work [15].

Theorem 4.1. *Under **(HF)** and **(HFD)**, we have the following estimate*

$$\mathbb{E}\Big[\sup_{t\in[0,T]}\big|X_t - X_t^\pi\big|^2\Big] \;\leq\; K|\pi|\;,$$

for a constant K which does not depend on π.

5 Convergence of the scheme for the BSDE

To provide error estimates for the scheme of the BSDE, we need an additional regularity property for the coefficients g and f. We then introduce the following assumption.

(HBQD) There exists a constant K such that the function f satisfies

$$\begin{aligned}
|f(t,x,y,z,u)-f(t,x',y',z',u')| \;\leq\;& K_f\big[|x-x'|+|y-y'|+|u-u'|\big] \\
&+L_{f,z}(1+|z|+|z'|)|z-z'|\;,
\end{aligned}$$

for all $(x,x',y,y',z,z',u,u')\in\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2$.

5.1 A uniform bound for Z^0 and Z^1

We present here a uniform bound for the processes Z^0 and Z^1. To this end, we need the following assumption.

(Hβ) The function β is differentiable w.r.t. x and there exists a constant $K_\nabla > 0$ such that

$$1+\nabla\beta(t,x) \;\geq\; K_\nabla\;, \quad (t,x)\in[0,T]\times\mathbb{R}\,.$$

Proposition 5.1. *Suppose that **(HF)**, **(HFD)**, **(HBQ)**, **(HBQD)** and **(Hβ)** hold and that g is Lipschitz with Lipschitz constant K_g. Then, there exists a version of $Z^1(\theta)$ such that*

$$|Z_t^1(\theta)| \;\leq\; e^{(2L_a+K_f)T}(K_g+TK_f)M_\sigma\;, \quad \theta\leq t\leq T\;.$$

Proof. In this proof, we omit the variable θ to improve readability. In the case where b, f and g are differentiable w.r.t. x, y and z, (X^1, Y^1, Z^1) is also differentiable w.r.t. x and we can see that

$$\nabla^\theta X_t^1 \;=\; 1 + \int_\theta^t \nabla b(s, X_s^1) \nabla^\theta X_s^1 ds \;, \quad \theta \leq t \leq T \;, \tag{5.1}$$

and

$$\begin{aligned} \nabla^\theta Y_t^1 \;=\; & \nabla^\theta g(X_T^1) \nabla^\theta X_T^1 - \int_t^T \nabla^\theta Z_s^1 dW_s \\ & + \int_t^T \nabla^\theta f(s, X_s^1, Y_s^1, Z_s^1, 0) \big(\nabla^\theta X_s^1, \nabla^\theta Y_s^1, \nabla^\theta Z_s^1 \big) ds \;, \end{aligned} \tag{5.2}$$

for $t \in [\theta, T]$. Since ∇b is uniformly bounded, we get from (5.1) and Gronwall's lemma

$$|\nabla^\theta X_t^1| \;\leq\; e^{L_a T} \;, \quad \theta \leq t \leq T \;. \tag{5.3}$$

Using **(Hβ)**, (5.3), we get

$$|(\nabla^\theta X_t^1)^{-1}| \;\leq\; e^{L_a T} \;, \quad \theta \leq t \leq T \;. \tag{5.4}$$

Applying Itô's formula, we get

$$\begin{aligned} e^{\int_\theta^t \nabla_y^\theta f(s, X_s^1, Y_s^1, Z_s^1, 0) ds} \nabla^\theta Y_t^1 \;=\; & e^{\int_\theta^T \nabla_y^\theta f(s, X_s^1, Y_s^1, Z_s^1, 0) ds} \nabla^\theta g(X_T^1) \nabla^\theta X_T^1 \\ & + \int_t^T e^{\int_\theta^s \nabla_y^\theta f(r, X_r^1, Y_r^1, Z_r^1, 0) dr} \nabla_x^\theta f(s, X_s^1, Y_s^1, Z_s^1, 0) \nabla^\theta X_s^1 ds \\ & - \int_t^T e^{\int_\theta^s \nabla_y^\theta f(r, X_r^1, Y_r^1, Z_r^1, 0) dr} \nabla^\theta Z_s^1 d\tilde{W}_s^1 \;, \end{aligned} \tag{5.5}$$

where $d\tilde{W}_s^1 = dW_s - \nabla_z^\theta f(s, X_s^1, Y_s^1, Z_s^1, 0) ds$. From **(HBQD)**, we have

$$\begin{aligned} \Big\| \int_\theta^\cdot \nabla_z^\theta f(s, X_s^1, Y_s^1, Z_s^1, 0) dW_s \Big\|_{BMO}^2 \;\leq\; & K \Big(1 + \sup_{\vartheta \in [\theta, T]} \mathbb{E} \Big[\int_\vartheta^T |Z_s^1|^2 ds \Big| \mathcal{F}_\vartheta \Big] \Big) \\ \leq\; & K \Big(1 + \Big\| \int_\theta^\cdot Z_s^1 dW_s \Big\|_{BMO}^2 \Big) \\ <\; & \infty \;, \end{aligned}$$

where the last inequality comes from the fact that under **(HF)**, **(HFD)**, **(HBQ)** and **(HBQD)** $\int_\theta^\cdot Z_s^1 dW_s$ belongs to the space BMO (see METTRE REF). Therefore, we can apply Girsanov's theorem: there exists a probability measure $\mathbb{Q}^1$ under which $\tilde{W}^1$ is a Brownian motion. We then get from (5.5)

$$\begin{aligned} e^{\int_\theta^t \nabla_y f(s, X_s^1, Y_s^1, Z_s^1, 0) ds} \nabla^\theta Y_t^1 \;=\; & \mathbb{E}_{\mathbb{Q}^1} \Big[e^{\int_\theta^T \nabla_y^\theta f(s, X_s^1, Y_s^1, Z_s^1, 0) ds} \nabla^\theta g(X_T^1) \nabla^\theta X_T^1 \\ & + \int_t^T e^{\int_\theta^s \nabla_y^\theta f(r, X_r^1, Y_r^1, Z_r^1, 0) dr} \nabla_x^\theta f(s, X_s^1, Y_s^1, Z_s^1, 0) \nabla^\theta X_s^1 ds \Big| \mathcal{F}_t \Big] \;. \end{aligned}$$

This last inequality, **(HBQD)** and (5.3) give

$$|\nabla^\theta Y_t^1| \leq e^{(L_a+K_f)T}(K_g+TK_f)\ , \quad \theta \leq t \leq T\ . \tag{5.6}$$

Moreover, using Malliavin calculus, we have the classical representation of the process Z^1 given by $\nabla^\theta Y^1(\nabla^\theta X^1)^{-1}\sigma(.)$. We therefore obtain from (5.4) and (5.6)

$$|Z_t^1| \leq e^{(2L_a+K_f)T}(K_g+TK_f)M_\sigma \quad a.s.$$

When b, f and g are not differentiable, we can also prove the result by a standard approximation and stability results for BSDEs with linear growth. □

Some useful estimates of Z^0

Proposition 5.2. *Suppose that **(HFQD)** and **(HBQD)** hold and that g is Lipschitz with Lipschitz constant K_g. Then, there exists a version of Z^0 such that*

$$|Z_t^0| \leq e^{2(K_f+L_a)T}(K_g+K_fT)\big(1+TK_fe^{K_fT}(1+L_ae^{L_aT})\big)M_\sigma\ , \quad 0 \leq t \leq T\ .$$

Proof. To simplify the notations we write $h(t,x,y,z)$ instead of $f(t,x,y,z,Y_t^1(t)-y)$. Firstly, we suppose that b, g and f are differentiable w.r.t. x, y, z and u. Then, (X^0,Y^0,Z^0) is differentiable w.r.t. x and $(\nabla X^0,\nabla Y^0,\nabla Z^0)$ is solution of

$$\nabla X_t^0 = 1+\int_0^t \nabla b(s,X_s^0)ds\nabla X_s^0 ds\ , \tag{5.7}$$

and

$$\begin{aligned}\nabla Y_t^0 = {} & \nabla g(X_T^0)\nabla X_T^0+\int_t^T\Big(\nabla_x h(s,X_s^0,Y_s^0,Z_s^0)\nabla X_s^0+\nabla_y h(s,X_s^0,Y_s^0,Z_s^0)\nabla Y_s^0\\ & +\nabla_z h(s,X_s^0,Y_s^0,Z_s^0)\nabla Z_s^0+\nabla_u h(s,X_s^0,Y_s^0,Z_s^0)(\nabla Y_s^1(s)-\nabla Y_s^0)\Big)ds\\ & -\int_t^T \nabla Z_s^0 dW_s\ .\end{aligned} \tag{5.8}$$

Thanks to usual transformations on the BSDEs we obtain

$$\begin{aligned}\nabla Y_t^0 = {} & e^{\int_t^T(\nabla_y-\nabla_u)h(s,X_s^0,Y_s^0,Z_s^0)ds}\nabla g(X_T^0)\nabla X_T^0\\ & +\int_t^T e^{\int_t^s(\nabla_y-\nabla_u)h(r,X_r^0,Y_r^0,Z_r^0)dr}\nabla_x h(s,X_s^0,Y_s^0,Z_s^0)\nabla X_s^0 ds\\ & +\int_t^T e^{\int_t^s(\nabla_y-\nabla_u)h(r,X_r^0,Y_r^0,Z_r^0)dr}\nabla_u h(s,X_s^0,Y_s^0,Z_s^0)(\nabla Y_s^1(s)-\nabla Y_s^0)ds\\ & -\int_t^T e^{\int_t^s(\nabla_y-\nabla_u)h(r,X_r^0,Y_r^0,Z_r^0)dr}\nabla Z_s^0 d\tilde{W}_s^0\end{aligned}$$

where $d\tilde{W}_s^0 = dW_s-\nabla_z h(s,X_s^0,Y_s^0,Z_s^0)ds$.

Since $Z^0 * W$ belongs to the space of BMO martingales, there exists a probability $\mathbb{Q}^0$ under which $\tilde{W}^0$ is a Brownian motion. Then,

$$\begin{aligned}\nabla Y_t^0 &= \mathbb{E}_{\mathbb{Q}^0}\Big[e^{\int_0^T(\nabla_y-\nabla_u)h(s,X_s^0,Y_s^0,Z_s^0)ds}\nabla g(X_T^0)\nabla X_T^0 \\ &\quad+\int_t^T e^{\int_0^s(\nabla_y-\nabla_u)h(r,X_r^0,Y_r^0,Z_r^0)dr}\nabla_x h(s,X_s^0,Y_s^0,Z_s^0)\nabla X_s^0 ds \\ &\quad+\int_t^T e^{\int_0^s(\nabla_y-\nabla_u)h(r,X_r^0,Y_r^0,Z_r^0)dr}\nabla_u h(s,X_s^0,Y_s^0,Z_s^0)\nabla Y_s^1(s)ds\Big|\mathcal{F}_t\Big] .\end{aligned}$$

Since ∇b is uniformly bounded, we get from (5.7) and Gronwall's lemma

$$|\nabla X_t^0| \leq e^{L_a T} , \quad 0\leq t\leq T , \tag{5.9}$$

we also prove that $|\nabla Y_t^1(t)| \leq (1+L_a e^{L_aT})e^{(L_a+K_f)T}(K_g+TK_f)$. Using these two inequalities we get

$$|\nabla Y_t^0| \leq e^{(2K_f+L_a)T}(K_g+K_fT)\big(1+TK_fe^{K_fT}(1+L_ae^{L_aT})\big) .$$

Moreover, thanks to the Malliavin calculus, it is classical to show that a version of Z^0 is given by $\nabla Y^0(\nabla X^0)^{-1}\sigma(.)$. So we obtain

$$|Z_t^0| \leq e^{2(K_f+L_a)T}(K_g+K_fT)\big(1+TK_fe^{K_fT}(1+L_ae^{L_aT})\big)M_\sigma \quad a.s.$$

because $|(\nabla X_t^0)^{-1}| \leq e^{L_aT}$.

When b, g and f are not differentiable, we can also prove the result by a standard approximation and stability results for BSDEs with linear growth. □

5.2 A time dependent bound Z^0 and Z^1

We also get a second bound for $Z^1(\theta)$ which depends on the time t but for that we introduce two alternative assumptions.

Assumption 5.1. b is differentiable w.r.t. x, and σ and β are differentiable w.r.t. t. There exists $\lambda\in\mathbb{R}_+$ such that

$$|\sigma(t)|\big|\sigma(t)\nabla b(t,x)-\sigma'(t)\big| \leq \lambda|\sigma(t)|^2 .$$

Assumption 5.2. σ is invertible and $\forall\, t\in[0,T]$, $|\sigma(t)^{-1}|\leq M_{\sigma^{-1}}$.

Proposition 5.3. *Suppose that **(HFQD)** and **(HBQD)** hold and that Assumptions 5.1 or 5.2 hold. Moreover, suppose that g is lower (or upper) semi-continuous. Then, there exists a version of $Z^1(\theta)$ and there exists a constant $K_{z^1}\in\mathbb{R}_+$ that depends only on T, M_g, K_q, K_f and $L_{f,z}$ such that*

$$|Z_t^1(\theta)| \leq K_{z^1}\big(1+(T-t)^{-1/2}\big) , \qquad \theta\leq t\leq T .$$

Proof. In a first time, we will suppose that Assumption 5.1 holds and that f and g are differentiable w.r.t. x, y and z. Then, we have

$$F_t \;=\; F_T - \int_t^T e^{\int_\theta^s \nabla_y^\theta f(u,X_u^1,Y_u^1,Z_u^1,0)du} \nabla^\theta Z_s^1 d\tilde{W}_s \ ,$$

where

$$F_t \;:=\; e^{\int_\theta^t \nabla_y^\theta f(u,X_u^1,Y_u^1,Z_u^1,0)du} \nabla^\theta Y_t^1 + \int_\theta^t e^{\int_\theta^s \nabla_y^\theta f(u,X_u^1,Y_u^1,Z_u^1,0)du} \nabla_x^\theta f(u, X_u^1, Y_u^1, Z_u^1, 0) \nabla^\theta X_s^1 ds \ .$$

Now we define

$$\begin{aligned}
\alpha_t &:= \int_\theta^t e^{\int_\theta^s \nabla_y^\theta f(u,X_u^1,Y_u^1,Z_u^1,0)du} \nabla_x^\theta f(u, X_u^1, Y_u^1, Z_u^1, 0) \nabla^\theta X_s^1 ds (\nabla^\theta X_s^1)^{-1} \sigma(t) \ , \\
\tilde{Z}_t &:= F_t (\nabla^\theta X_s^1)^{-1} \sigma(t) = e^{\int_\theta^t \nabla_y^\theta f(u,X_u^1,Y_u^1,Z_u^1,0)du} Z_t + \alpha_t \ , \\
\tilde{F}_t &:= e^{\lambda t} F_t (\nabla^\theta X_s^1)^{-1} \ .
\end{aligned}$$

Since $d\nabla^\theta X_t^1 = \nabla^\theta b(t, X_t^1) \nabla^\theta X_t^1 dt$, then $d(\nabla^\theta X_t^1)^{-1} = -(\nabla^\theta X_t^1)^{-1} \nabla^\theta b(t, X_t^1) dt$ and thanks to Itô's formula

$$d\tilde{Z}_t \;=\; dF_t (\nabla^\theta X_t^1)^{-1} \sigma(t) - F_t (\nabla^\theta X_t^1)^{-1} \nabla^\theta b(t, X_t^1) \sigma(t) dt + F_t (\nabla^\theta X_t^1)^{-1} \sigma(t) \ ,$$

and

$$d(e^{\lambda t} \tilde{Z}_t) \;=\; \tilde{Z}_t [\lambda - \nabla^\theta b(t, X_t^1)] \sigma(t) dt + \tilde{F}_t \sigma'(t) dt + e^{\lambda t} dF_t (\nabla^\theta X_t^1)^{-1} \sigma(t) \ .$$

Finally,

$$d|e^{\lambda t} \tilde{Z}_t|^2 \;=\; d\langle M_t \rangle + 2\big[\lambda |\tilde{F}_t \sigma(t)|^2 - \tilde{F}_t^2 \sigma(t) [\sigma(t) \nabla^\theta b(t, X_t^1) - \sigma'(t)]\big] dt + dM_t^*$$

with $M_t := \int_0^t e^{\lambda s} dF_s (\nabla^\theta X_s^1)^{-1} \sigma(s)$ and M^* is a $\mathbb{Q}^1$-martingale. Thanks to Assumption 5.1 we are able to conclude that $|e^{\lambda t} \tilde{Z}_t|^2$ is a $\mathbb{Q}^1$-submartingale. Hence

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}^1}\Big[\int_t^T e^{2\lambda s} |\tilde{Z}_s|^2 ds \Big| \mathcal{F}_t \Big] &\geq e^{2\lambda t} |\tilde{Z}_t|^2 (T - t) \\
&\geq e^{2\lambda t} \Big| e^{\int_\theta^t \nabla_y^\theta f(s,X_s^1,Y_s^1,Z_s^1,0)ds} Z_t + \alpha_t \Big|^2 (T - t)
\end{aligned}$$

which implies

$$\begin{aligned}
|Z_t^1|^2 (T - t) &= e^{-2\lambda t} e^{-2\int_\theta^t \nabla_y^\theta f(s,X_s^1,Y_s^1,Z_s^1,0)ds} e^{\lambda t} \Big| e^{\int_\theta^t \nabla_y^\theta f(s,X_s^1,Y_s^1,Z_s^1,0)ds} Z_t^1 + \alpha_t - \alpha_t \Big|^2 (T - t) \\
&\leq K \Big(e^{2\lambda t} \Big| e^{\int_\theta^t \nabla_y^\theta f(s,X_s^1,Y_s^1,Z_s^1,0)ds} Z_t^1 + \alpha_t \Big|^2 + 1 \Big) (T - t) \\
&\leq K \Big(\mathbb{E}_{\mathbb{Q}^1}\Big[\int_t^T e^{2\lambda s} |\tilde{Z}_s|^2 ds \Big| \mathcal{F}_t \Big] + (T - t) \Big)
\end{aligned}$$

with K a constant that only depends on T, K_f, L_a, K_a and λ and not on θ. Moreover, we have

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}^1}\Big[\int_{t^T} e^{2\lambda s}|\tilde{Z}_s|^2 ds\Big|\mathcal{F}_t\Big] &\leq K\mathbb{E}_{\mathbb{Q}^1}\Big[\int_t^T \big(|Z_s^1|^2+|\alpha_s|^2\big)ds\Big|\mathcal{F}_t\Big]\\ &\leq K\big(||Z^1||^2_{BMO(\mathbb{Q}^1)}+(T-t)\big)\end{aligned}$$

But $||Z^1||_{BMO(\mathbb{Q}^1)}$ does not depend on K_g because (Y^1, Z^1) is a solution of the following BSDE:

$$Y_t^1 = g(X_T^1)+\int_t^T \big(f(s,X_s^1,Y_s^1,Z_s^1,0)-Z_s^1\nabla_z^\theta f(s,X_s^1,Y_s^1,Z_s^1,0)\big)ds-\int_t^T Z_s^1 d\tilde{W}_s\ .$$

Finally $|Z_t^1| \leq K(1+(T-t)^{-1/2})$ a.s.

When σ is invertible, the inequality of Assumption 5.1 is verified with $\lambda := M_{\sigma^{-1}}(K_aL_a+K_t)$. Since λ does not depend on ∇b and σ', we can prove the result when $b(t,.)$ and σ are not differentiable by a standard approximation and stability results for BSDEs with linear groath. So, we are allowed to replace Assumption 5.1 by Assumption 5.2. □

We get another bound for Z^0 which depends on the time t.

Proposition 5.4. *Suppose that **(HFQD)**, **(HBQD)** hold and that Assumptions 5.1 or 5.2 hold. Moreover, suppose that g is lower (or upper) semi-continuous. Then, there exists a version of Z^0 and there exists a constant $K_{z^0} > 0$ that depends only on T, M_g, K_q, K_f and $L_{f,z}$ such that,*

$$|Z_t^0| \leq K_{z^0}\big(1+(T-t)^{-1/2}\big)\ , \quad 0 \leq t \leq T\ .$$

Proof. In a first time, we will suppose that Assumption 5.1 holds and that f, g are differentiable w.r.t. x, y, z and u. Then, (Y^0, Z^0) is differentiable w.r.t. x and $(\nabla Y^0, \nabla Z^0)$ is the solution of BSDE (5.8). We denote

$$\begin{aligned}F_t := &\ e^{\int_0^t(\nabla_y-\nabla_u)h(s,X_s^0,Y_s^0,Z_s^0)ds}\nabla Y_t^0+\int_0^t e^{\int_0^s(\nabla_y-\nabla_u)h(r,X_r^0,Y_r^0,Z_r^0)dr}\\ &(\nabla_x h(s,X_s^0,Y_s^0,Z_s^0)\nabla X_s^0+\nabla_u h(s,X_s^0,Y_s^0,Z_s^0)\nabla Y_s^1(s))ds\ .\end{aligned}$$

We can write

$$F_t = F_T-\int_t^T e^{\int_0^s(\nabla_y-\nabla_u)h(r,X_r^0,Y_r^0,Z_r^0)dr}\nabla Z_s^0 d\tilde{W}_s\ .$$

Now, we define

$$\begin{aligned}\alpha_t &:= \int_0^t e^{\int_0^r(\nabla_y-\nabla_u)h(s,X_s^0,Y_s^0,Z_s^0)ds}\big(\nabla_x h(r,X_r^0,Y_r^0,Z_r^0)\nabla X_r^0+\nabla_u h(r,X_r^0,Y_r^0,Z_r^0)\nabla Y_r^1(r)\big)dr(\nabla X_t^0)^{-1}\sigma(t)\ ,\\ \tilde{Z}_t &:= F_t(\nabla X_t^0)^{-1}\sigma(t) = e^{\int_0^t(\nabla_y-\nabla_u)h(s,X_s^0,Y_s^0,Z_s^0)ds}Z_t^0+\alpha_t\ ,\ a.s.\ ,\\ \tilde{F}_t &:= e^{\lambda t}F_t(\nabla X_t^0)^{-1}\ .\end{aligned}$$

Thanks to Itô's formula

$$d\tilde{Z}_t \;=\; (\nabla X_t^0)^{-1}\sigma(t)dF_t - F_t(\nabla X_t^0)^{-1}\nabla b(t,X_t^0)\sigma(t)dt + F_t(\nabla X_t^0)^{-1}\sigma'(t)dt\;,$$

and

$$d(e^{\lambda t}\tilde{Z}_t) \;=\; \tilde{F}_t(\lambda - \nabla b(t,X_t^0))\sigma(t)dt + \tilde{F}_t\sigma'(t)dt + e^{\lambda t}(\nabla X_t^0)^{-1}\sigma(t)dF_t\;.$$

Finally,

$$d\big|e^{\lambda t}\tilde{Z}_t\big|^2 \;=\; d\langle M\rangle_t + 2\Big[\lambda\big|\tilde{F}_t\sigma(t)\big|^2 - \tilde{F}_t^2\sigma_t[\sigma(t)\nabla b(t,X_t^0) - \sigma'(t)]\Big]dt + M_t^*\;,$$

with $M_t := \int_0^t e^{\lambda s}dF_s(\nabla X_s^0)^{-1}\sigma(s)$ and M^* a $\mathbb{Q}^0$-martingale. Thanks to Assumption 5.1, we are able to conclude that $|e^{\lambda t}\tilde{Z}_t|^2$ is a $\mathbb{Q}^0$-martingale. Therefore, we obtain as in [22] that $|Z_t^0| \le K(1+(T-t)^{-1/2})$ a.s.

When f is not differentiable and g is only Lipschitz, we can prove the result by a standard approximation and stability results for BSDEs. □

5.3 Approximation of FBSDE (2.3)-(2.4)

An approximation of the quadratic BSDE

Throughout the sequel, we approximate BSDE (2.3)-(2.4) by another one. Let $(Y^{N,\epsilon}(\theta), Z^{N,\epsilon}(\theta))$ be the solution of the BSDE

$$\begin{aligned} Y_t^{N,\epsilon}(\theta) \;&=\; g_N(X_T^1(\theta)) + \int_t^T f^\epsilon\big(s, X_s^1(\theta), Y_s^{N,\epsilon}(\theta), Z_s^{N,\epsilon}(\theta)\big)ds \\ &\quad - \int_t^T Z_s^{N,\epsilon}(\theta)dW_s\;, \qquad \theta \le t \le T\;, \end{aligned} \tag{5.10}$$

we recall that

$$f^\epsilon(s,x,y,z) \;=\; \mathbb{1}_{s\le T-\epsilon}f(s,x,y,z,0) + \mathbb{1}_{s>T-\epsilon}f(s,x,y,0,0)\;,$$

and g_N a Lipschitz approximation of g with Lipschitz constant N. f^ϵ verifies **(HBQD)** with the same constants. Moreover, we can apply Proposition 5.3 to obtain an upper bound for $Z^{N,\epsilon}(\theta)$.

Proposition 5.5. *Let us assume that **(HFQD)**, **(HBQD)** and Assumptions 5.1 or 5.2 hold. There exists a version of $Z^{N,\epsilon}(\theta)$ and there exists a constant $M_{z^1} \in \mathbb{R}_+$ that does not depend on N and ϵ such that,*

$$|Z_t^{N,\epsilon}(\theta)| \;\le\; M_{z^1}\big[\big(1+(T-t)^{-1/2}\big)\wedge(N+1)\big]\;, \quad \theta\le t\le T\;.$$

Thanks to BMO tools, we have a stability result for quadratic BSDEs: Mettre une ref

Proposition 5.6. *Let us assume that **(HFQD)** and **(HBQD)** hold. There exists a constant K that does not depend on N, ϵ and θ such that*

$$\mathbb{E}\Big[\sup_{t\in[\theta,T]}\big|Y_t^{N,\epsilon}(\theta)-Y_t^1(\theta)\big|^2\Big]+\mathbb{E}\Big[\int_\theta^T\big|Z_t^{N,\epsilon}(\theta)-Z_t^1(\theta)\big|^2dt\Big] \;\leq\; K(e_1(\theta,N)+e_2(\theta,N,\epsilon))\,,$$

with

$$e_1(\theta,N) \;:=\; \mathbb{E}\big[\big|g_N(X_T^1(\theta))-g(X_T^1(\theta))\big|^{2q}\big]^{1/q}\,,$$

$$e_2(\theta,N,\epsilon) \;:=\; \mathbb{E}\Big[\Big(\int_{T-\epsilon\vee\theta}^T\Big|f\big(t,X_t^1(\theta),Y_t^{N,\epsilon}(\theta),Z_t^{N,\epsilon}(\theta),0\big)-f\big(t,X_t^1(\theta),Y_t^{N,\epsilon}(\theta),0,0\big)\Big|dt\Big)^{2q}\Big]^{1/q}\,,$$

and q defined in Theorem **??**. *Quel Th*

The aim of our work is to study the error of discretization

$$e(\theta,N,\epsilon,\pi) \;:=\; \sup_{\theta\leq t\leq T}\mathbb{E}\Big[\big|Y_{\pi(t)}^{1,\pi}(\pi(\theta))-Y_t^1(\theta)\big|^2\Big]+\mathbb{E}\Big[\int_\theta^T\big|Z_{\pi(t)}^{1,\pi}(\pi(\theta))-Z_t(\theta)\big|^2dt\Big]\,,$$

where $(Y^{1,\pi},Z^{1,\pi})$ is defined by (3.11).

It is easy to see that there exists a constant K such that

$$e(\theta,N,\epsilon,\pi) \;\leq\; K\big(e_1(\theta,N)+e_2(\theta,N,\epsilon)+e_3(\theta,N,\epsilon,\pi)+e_4(\theta,N,\epsilon,\pi)\big)\,,$$

with $e_1(\theta,N)$ and $e_2(\theta,N,\epsilon)$ defined in Proposition 5.6 and

$$e_3(\theta,N,\epsilon,\pi) \;:=\; \sup_{t\in[\theta,T]}\mathbb{E}\Big[\big|Y_t^{N,\epsilon}(\pi(\theta))-Y_t^{N,\epsilon}(\theta)\big|^2\Big]+\mathbb{E}\Big[\int_\theta^T\big|Z_t^{N,\epsilon}(\pi(\theta))-Z_t^{N,\epsilon}(\theta)\big|^2dt\Big]\,,$$

$$e_4(\theta,N,\epsilon,\pi) \;:=\; \sup_{t\in[\theta,T]}\mathbb{E}\Big[\big|Y_t^{N,\epsilon}(\pi(\theta))-Y_{\pi(t)}^{1,\pi}(\pi(\theta))\big|^2\Big]+\mathbb{E}\Big[\int_\theta^T\big|Z_t^{N,\epsilon}(\pi(\theta))-Z_{\pi(t)}^{1,\pi}(\pi(\theta))\big|^2dt\Big]\,.$$

Study of the time approximation error $e_4(\theta,N,\epsilon,\pi)$

We set $\epsilon=Tn^{-a}$ and $N=n^b$, with $a,b\in\mathbb{R}_+^*$ two parameters. Firstly, we give a convergence result for the Euler scheme.

Proposition 5.7. *Assume **(HFQD)** holds. Then, there exists a constant K that does not depend on n and θ, such that*

$$\sup_{t\in[\theta,T]}\mathbb{E}\big[\big|X_t^1(\theta)-X_{\pi(t)}^{1,\pi}(\pi(\theta))\big|^2\big] \;\leq\; K\frac{\ln n}{n}\,.$$

Proof. As in the proof of Theorem 4.1, we have that

$$\sup_{t\in[\theta,T]}\mathbb{E}\big[\big|X_t^1(\theta)-X_{\pi(t)}^{1,\pi}(\pi(\theta))\big|^2\big] \;\leq\; K\sup_{1\leq i\leq 2n}\Delta t_i^\pi \;=\; K\Delta t_0^\pi\,.$$

But $\Delta t_0^\pi=T(1-n^{-a/n})\leq C\frac{\ln(n)}{n}$, so the proof is ended. □

For the sequel, we need an extra assumption.

Assumption 5.3. There exists a positive constant $K_{f,t}$ such that $\forall\ t, t' \in [0,T]$, $\forall\ x \in \mathbb{R}$, $\forall\ y \in \mathbb{R}$, $\forall\ z \in \mathbb{R}$, $\forall\ u \in \mathbb{R}$

$$\big|f(t,x,y,z,u) - f(t',x,y,z,u)\big| \ \leq\ K_{f,t}|t-t'|^{1/2} .$$

By adapting the proof of [22], we get the BSDE approximation.

Proposition 5.8. *Assume that* ***(HFQD)***, ***(HBQD)***, *Assumptions 5.3 and 5.1 or 5.2 hold. Then, for all $\eta > 0$, there exists a constant K that does not depend on N, ϵ, n and θ, such that*

$$\sup_{t\in[\theta,T]} \mathbb{E}\Big[\big|Y^{1,\pi}_{\pi(t)}(\pi(\theta)) - Y^{N,\epsilon}_t(\pi(\theta))\big|^2\Big] + \mathbb{E}\Big[\int_\theta^T \big|Z^{1,\pi}_{\pi(t)}(\pi(\theta)) - Z^{N,\epsilon}_t(\pi(\theta))\big|^2 dt\Big]$$
$$\leq \frac{K}{n^{1-2b-\mathcal{K}a}} + \frac{K}{n^{1+\eta-4b}} ,$$

with $\mathcal{K} = 4(1+\eta)L^2_{f,z}M^2_{z^1}$.

Proof. It is sufficient to give an upper bound of

$$\sup_{t\in[\theta,T]} \mathbb{E}\Big[\big|Y^{N,\epsilon}_t(\pi(\theta)) - Y^{N,\epsilon}_{\pi(t)}(\pi(\theta))\big|^2\Big] ,$$

because the upper bound of

$$\sup_{t\in[\theta,T]} \mathbb{E}\Big[\big|Y^{1,\pi}_{\pi(t)}(\pi(\theta)) - Y^{N,\epsilon}_{\pi(t)}(\pi(\theta))\big|^2\Big] + \mathbb{E}\Big[\int_\theta^T \big|Z^{1,\pi}_{\pi(t)}(\pi(\theta)) - Z^{N,\epsilon}_t(\pi(\theta))\big|^2 dt\Big]$$

is given by [22]. Using BSDE (5.10), we get

$$Y^{N,\epsilon}_t(\pi(\theta)) - Y^{N,\epsilon}_{\pi(t)}(\pi(\theta)) \ =\ \int_{\pi(t)}^t f^\epsilon\big(s, X^1_s(\pi(\theta)), Y^{N,\epsilon}_s(\pi(\theta)), Z^{N,\epsilon}_s(\pi(\theta))\big)ds - \int_{\pi(t)}^t Z^{N,\epsilon}_s(\pi(\theta))dW_s .$$

Since f^ϵ satisfies **(HBQD)** and using Proposition 5.1, we can see that there exists a constant K such that

$$\sup_{t\in[\theta,T]} \mathbb{E}\Big[\big|Y^{N,\epsilon}_t(\pi(\theta)) - Y^{N,\epsilon}_{\pi(t)}(\pi(\theta))\big|^2\Big] \ \leq\ K(1+n^{4b}) \sup_{1\leq i\leq 2n} \Delta t^\pi_i .$$

□

Study of the global error $e(\theta, N, \epsilon, \pi)$
Let us study errors $e_1(\theta,N)$, $e_2(\theta,N,\epsilon)$ and $e_3(\theta,N,\epsilon,\pi)$.

Proposition 5.9. *Let us assume that* ***(HFQD)*** *and* ***(HBQD)*** *hold. There exists a constant $K > 0$ such that*

$$e_2(\theta,N,\epsilon) \ \leq\ \frac{K}{n^{2a-4b}} .$$

Proposition 5.10. *We assume that* **(HFQD)** *holds and g is α-Hölder. Then, there exists a constant $K > 0$ such that*

$$e_1(\theta, N) \leq \frac{K}{n^{\frac{2b\alpha}{1-\alpha}}} .$$

The proofs of these propositions are given in [22].

We now give an upper bound for the error $e_3(\theta, N, \epsilon, \pi)$.

Proposition 5.11. *Let us assume that* **(HFQD)** *and* **(HBQD)** *hold. Then, for all $\eta > 0$, there exists a constant $K > 0$ such that*

$$e_3(\theta, N, \epsilon, \pi) \leq \frac{K}{n^{1-K'a+\eta}} ,$$

with $K' = 4L_f^2 M_z^2$.

Proof. We denote $\eta_t := Y_t^{N,\epsilon}(\theta) - Y_t^{N,\epsilon}(\pi(\theta))$ and $\mu_t := |Y_t^{N,\epsilon}(\theta) - Y_t^{N,\epsilon}(\pi(\theta))|^2$. Using Itô's formula and with **(HFQD)** and **(HBQD)**, we get

$$\begin{aligned}
\mathbb{E}[\mu_t] \leq\ & \mathbb{E}[\mu_T] - \mathbb{E}\Big[\int_t^T \big|Z_s^{N,\epsilon}(\theta) - Z_s^{N,\epsilon}(\pi(\theta))\big|^2 ds\Big] + (1+K_f)\mathbb{E}\Big[\int_t^T \mu_s ds\Big] \\
& + K_f^2 \mathbb{E}\Big[\int_t^T \big|X_s^1(\theta) - X_s^1(\pi(\theta))\big|^2 ds\Big] \\
& + 2\mathbb{E}\Big[\int_t^{T-\epsilon} |\eta_s|\big(K_f + 2L_{f,z}M_{z^1}(1+(T-s)^{-1/2})\big)\big|Z_s^{N,\epsilon}(\theta) - Z_s^{N,\epsilon}(\pi(\theta))\big| ds\Big] ,
\end{aligned}$$

we can see that

$$\begin{aligned}
\mathbb{E}[\mu_t] \leq\ & K_g^2 \mathbb{E}\big[\big|X_T^1(\theta) - X_T^1(\pi(\theta))\big|^2\big] + K_f^2 \mathbb{E}\Big[\int_t^T \big|X_s^1(\theta) - X_s^1(\pi(\theta))\big|^2 ds\Big] \\
& + \mathbb{E}\Big[\int_t^T \Big(1 + K_f + 2K_f^2 + \frac{4L_{f,z}^2 M_{z^1}^2}{T-s}\mathbb{1}_{s\leq T-\epsilon}\Big)\mu_s ds\Big] .
\end{aligned}$$

Using Proposition 5.7 and Gronwall's Lemma, we get

$$\mathbb{E}\big[\big|Y_t^{N,\epsilon}(\theta) - Y_t^{N,\epsilon}(\pi(\theta))\big|^2\big] \leq K \frac{\ln n}{n^{1-4L_{f,z}^2 M_{z^1}^2 a}} .$$

We also get

$$\begin{aligned}
\mathbb{E}\Big[\int_t^T \big|Z_s^{N,\epsilon}(\theta) - Z_s^{N,\epsilon}(\pi(\theta))\big|^2 ds\Big] \leq\ & \mathbb{E}[\mu_T] + K\mathbb{E}\Big[\int_t^T \mu_s ds\Big] + K\mathbb{E}\Big[\int_t^T \big|X_s^1(\theta) - X_s^1(\pi(\theta))\big|^2 ds\Big] \\
& + 2\mathbb{E}\Big[\int_t^{T-\epsilon} |\eta_s|\big(K_f + 2L_{f,z}M_{z^1}(1+(T-s)^{-1/2})\big)\big|Z_s^{N,\epsilon}(\theta) - Z_s^{N,\epsilon}(\pi(\theta))\big| ds\Big] .
\end{aligned}$$

Using the inequality $2ab \leq a^2/\beta + \beta b^2$, there is exists a constant K such that

$$\begin{aligned}
\mathbb{E}\Big[\int_t^T \big|Z_s^{N,\epsilon}(\theta) - Z_s^{N,\epsilon}(\pi(\theta))\big|^2 ds\Big] \leq\ & K\Big\{\mathbb{E}[\mu_T] + \mathbb{E}\Big[\int_t^T \mu_s ds\Big] + \mathbb{E}\Big[\int_t^T \big|X_s^1(\theta) - X_s^1(\pi(\theta))\big|^2 ds\Big] \\
& + \mathbb{E}\Big[\int_t^{T-\epsilon} \big(K_f + 2L_{f,z}M_{z^1}(1+(T-s)^{-1/2})\big)^2 \mu_s ds\Big]\Big\} .
\end{aligned}$$

With the previous inequality, we get

$$\mathbb{E}\Big[\int_t^T \big|Z_s^{N,\epsilon}(\theta)-Z_s^{N,\epsilon}(\pi(\theta))\big|^2 ds\Big] \;\leq\; K\frac{(\ln n)^2}{n^{1-4L_{f,z}^2M_{z^1}^2 a}}\ .$$

□

Now we are able to gather all these errors.

Proposition 5.12. *We assume that* ***(HFQD)***, ***(HBQD)***, *Assumptions 5.3, and 5.1 or 5.2 hold. We assume also that g is α-Hölder. Then, for all $\eta > 0$, there exists a constant $K > 0$ that does not depend on n and θ such that*

$$e(\theta,N,\epsilon,\pi) \;\leq\; K\Big(\frac{1}{n^{\frac{2\alpha}{(2-\alpha)(2+K'')-2-2\alpha}}}+\frac{1}{n^{1-K'a+\eta}}\Big)\ ,$$

with $K' = 4L_{f,z}^2M_{z^1}^2$ and $K'' = 4(1+\eta)L_{f,z}^2M_{z^1}^2$.

Proof. Thanks to Propositions 5.8, 5.9, 5.10 and 5.11, we have

$$e(N,\epsilon,\pi) \;\leq\; \frac{K}{n^{\frac{2b\alpha}{1-\alpha}}}+\frac{K}{n^{2a-4b}}+\frac{K}{n^{1-K'a+\eta}}+\frac{K}{n^{1-2b-K''a}}\ .$$

Then, we only need to set $a := \frac{1+2b}{2+K''}$ and $b := \frac{1-\alpha}{(2-\alpha)(2+K'')-2+2\alpha}$ to obtain the result. □

5.4 Discretization of FBSDE (2.5)-(2.6)

Convergence of modified time discretization schemes for the BSDE

We now give an approximation scheme for the solution of FBSDE (2.5)-(2.6). We first approximate our quadratic BSDE by another one. We set $\epsilon \in]0,T[$ and $N \in \mathbb{N}$. Let $(Y^{N,\epsilon}, Z^{N,\epsilon})$ the solution of the BSDE

$$Y_t^{N,\epsilon} \;=\; g_N(X_T^0)+\int_t^T f^\epsilon(s,X_s^0,Y_s^{N,\epsilon},Z_s^{N,\epsilon})ds-\int_t^T Z_s^{N,\epsilon}dW_s\ , \qquad (5.11)$$

with

$$f^\epsilon(s,x,y,z) \;:=\; \mathbb{1}_{s\leq T-\epsilon}f(s,x,y,z,Y_s^1(s)-y)+\mathbb{1}_{s>T-\epsilon}f(s,x,y,0,Y_s^1(s)-y)\ ,$$

and g_N a Lipschitz approximation of g with Lipschitz constant N. f^ϵ verifies **(HBQD)** with the same constants as f. Since g_N is a Lipschitz function, $Z^{N,\epsilon}$ has a bounded version and BSDE (5.11) is a Lipschitz BSDE. Moreover, we can apply Proposition 5.4 to obtain an upper bound of $Z^{N,\epsilon}$.

Proposition 5.13. *Let us assume that* ***(HFQD)***, ***(HBQD)*** *and Assumptions 5.1 or 5.2 hold. There exists a version of $Z^{N,\epsilon}$ and there exists a constant $M_{z^0} > 0$ that does not depend on N and ϵ such that,*

$$|Z_t^{N,\epsilon}| \;\leq\; M_{z^0}\Big[\big(1+(T-t)^{-1/2}\big)\wedge(N+1)\Big]\ , \quad 0\leq t\leq T\ .$$

Proposition 5.14. *Let us assume that **(HFQD)** and **(HBQD)** hold. There exists a constant K that does not depend on N and ϵ such that*

$$\mathbb{E}\Big[\sup_{0\le t\le T}\big|Y_t^{N,\epsilon}-Y_t^0\big|^2\Big]+\mathbb{E}\Big[\int_0^T\big|Z_t^{N,\epsilon}-Z_t^0\big|^2dt\Big] \;\le\; K(e_1(N)+e_2(N,\epsilon))$$

with

$$e_1(N) \;:=\; \mathbb{E}\big[|g_N(X_T^0)-g(X_T^0)|^{2q}\big]^{1/q}\,,$$

$$e_2(N,\epsilon) \;:=\; \mathbb{E}\Big[\Big(\int_{T-\epsilon}^T\big|f\big(t,X_t^0,Y_t^{N,\epsilon},Z_t^{N,\epsilon},Y_t^1(t)-Y_t^{N,\epsilon}\big)-f\big(t,X_t^0,Y_t^{N,\epsilon},0,Y_t^1(t)-Y_t^{N,\epsilon}\big)\big|dt\Big)^{2q}\Big]^{1/q}\,,$$

and q defined in Theorem **??**. *Quel Thm?*

We now introduce another BSDE which approximate the Lipschitz BSDE (5.11) by replacing $Y_s^1(s)$ by $Y_{\pi(s)}^{1,\pi}(\pi(s))$:

$$\tilde{Y}_t^{N,\epsilon,\pi} \;=\; g_N(X_T^0)+\int_t^T f^{\pi,\epsilon}\big(s,X_s^0,\tilde{Y}_s^{N,\epsilon,\pi},\tilde{Z}_s^{N,\epsilon,\pi}\big)ds-\int_t^T\tilde{Z}_s^{N,\epsilon,\pi}dW_s\,, \qquad (5.12)$$

with $f^{\pi,\epsilon}(s,x,y,z)=f^{\epsilon}(s,x,y,z,Y_{\pi(s)}^{1,\pi}(\pi(s))-y)$.

Proposition 5.15. *Let us assume that **(HFQD)** and **(HBQD)** hold. There exists a constant K that does not depend on N and ϵ such that*

$$\mathbb{E}\Big[\sup_{0\le t\le T}\big|Y_t^{N,\epsilon}-\tilde{Y}_t^{N,\epsilon,\pi}\big|^2\Big]+\mathbb{E}\Big[\int_0^T\big|Z_t^{N,\epsilon}-\tilde{Z}_t^{N,\epsilon,\pi}\big|^2dt\Big] \;\le\; Ke_3(N,\epsilon,\pi)\,.$$

The aim of our work is to study the error of discretization

$$e(N,\epsilon,\pi) \;:=\; \sup_{0\le t\le T}\mathbb{E}\Big[\big|Y_{\pi(t)}^{0,\pi}-Y_t^0\big|^2\Big]+\mathbb{E}\Big[\int_0^T\big|Z_{\pi(t)}^{0,\pi}-Z_t^0\big|^2dt\Big]\,.$$

It is easy to see that

$$e(N,\epsilon,\pi) \;\le\; K\big(e_1(N)+e_2(N,\epsilon)+e_3(N,\epsilon,\pi)+e_4(N,\epsilon,\pi)\big)\,,$$

with $e_1(N)$ and $e_2(N,\epsilon)$ defined in Proposition 5.14 and $e_3(N,\epsilon,\pi)$ defined in Proposition 5.15, and

$$e_4(N,\epsilon,\pi) \;:=\; \sup_{0\le t\le T}\mathbb{E}\Big[\big|\tilde{Y}_t^{N,\epsilon,\pi}-Y_{\pi(t)}^{0,\pi}\big|^2\Big]+\mathbb{E}\Big[\int_0^T\big|\tilde{Z}_t^{N,\epsilon,\pi}-Z_{\pi(t)}^{0,\pi}\big|^2dt\Big]\,.$$

Study of the time approximation error $e_4(N,\epsilon,\pi)$

As in [22], we add an extra assumption

Assumption 5.4. There exists a positive constant K_t such that $\forall\, t,\ t' \in [0,T]$, $\forall\, x \in \mathbb{R}$, $\forall\, y \in \mathbb{R}$, $\forall\, z \in \mathbb{R}$,

$$\left|f^{\pi,\epsilon}(t,x,y,z) - f^{\pi,\epsilon}(t',x,y,z)\right| \;\leq\; K_{t,f}|t-t'|^{1/2}\,.$$

We set $\epsilon = Tn^{-a}$ and $N = n^b$, with $a, b \in \mathbb{R}_+^*$ two parameters.

Firstly, we give a convergence result for the Euler scheme.

Proposition 5.16. *Assume **(HFQD)** holds. Then, there exists a constant K that does not depend on n such that*

$$\sup_{0\leq t\leq T} \mathbb{E}\Big[\big|X_t^0 - X_{\pi(t)}^{0,\pi}\big|^2\Big] \;\leq\; K\frac{\ln n}{n}\,.$$

Since the generator $f^{\pi,\epsilon}$ is Lipschitz, we get the result of Theorem 4.8 of [22]:

$$\sup_{0\leq t\leq T} \mathbb{E}\Big[\big|\tilde{Y}_t^{N,\epsilon,\pi} - Y_{\pi(t)}^{0,\pi}\big|^2\Big] + \mathbb{E}\Big[\int_0^T \big|\tilde{Z}_t^{N,\epsilon,\pi} - Z_{\pi(t)}^{0,\pi}\big|^2 dt\Big] \;\leq\; \frac{K}{n^{1-2b-Ka}}\,,$$

with $K = 4(1+\eta)L_{f,z}^2 M_{z^0}^2$.

Study of the time approximation error $e_3(N,\epsilon,\pi)$

We denote $\eta_t = Y_t^{N,\epsilon} - \tilde{Y}_t^{N,\epsilon,\pi}$ and $\nu_t = |\eta_t|^2$. As in the proof of Proposition 5.11, there exists a constant K such that

$$\mathbb{E}[\nu_t] \;\leq\; K\int_t^T \mathbb{E}[\nu_s]ds + K\sup_{0\leq s\leq T}\mathbb{E}\Big[\big|Y_{\pi(s)}^{1,\pi}(\pi(s)) - Y_s^1(s)\big|^2\Big]\,.$$

Using Gronwall's lemma and Proposition 5.12, we get that there exists a constant K such that

$$e_3(N,\epsilon,\pi) \;\leq\; K\Big(\frac{1}{n^{\frac{2\alpha}{(2-\alpha)(2+K'')-2-2\alpha}}} + \frac{1}{n^{1-K'a+\eta}}\Big)\,,$$

with $K' = 4L_{f,z}^2 M_{z^0}^2$ and $K'' = 4(1+\eta)L_{f,z}^2 M_{z^0}^2$.

Study of the global error $e(N,\epsilon,\pi)$

From Proposition 4.9 and Proposition 4.10 from [22], we get that

$$e_2(N,\epsilon) \;\leq\; \frac{K}{n^{2a-4b}}\,,$$

and

$$e_1(N) \;\leq\; \frac{K}{n^{\frac{2b\alpha}{1-\alpha}}}\,,$$

with g is α-Hölder.

Now, we are able to gather all these errors.

Proposition 5.17. *We assume that* ***(HFQD)***, ***(HBQD)***, *Assumptions 5.3, and 5.1 or 5.2 hold. We assume also that g is α-Hölder. Then, for all $\eta > 0$, there exists a constant K that does not depend on n such that*

$$e(N,\epsilon,\pi) \quad \leq \quad K\Big(\frac{1}{n^{\frac{2\alpha}{(2-\alpha)(2+K'')-2-2\alpha}}} + \frac{1}{n^{1-K'a+\eta}}\Big) ,$$

with $K' = 4L_{f,z}^2 M_{z^0}$ and $K'' = 4(1+\eta)L_{f,z}^2 M_{z^0}$.

5.5 Approximation of the solution of FBSDE (2.1)-(2.2)

In this part, we give an approximation of the solution of FBSDE (2.1)-(2.2) by using the previous results.We introduce the following scheme for $i = 0, \dots, n$

$$\begin{aligned} Y_{t_i}^{\pi} &= Y_{t_i}^{0,\pi}\mathbb{1}_{t_i<\tau} + Y_{t_i}^{1,\pi}(\pi(\tau))\mathbb{1}_{t_i\geq\tau} \\ Z_{t_i}^{\pi} &= Z_{t_i}^{0,\pi}\mathbb{1}_{t_i\leq\tau} + Z_{t_i}^{1,\pi}(\pi(\tau))\mathbb{1}_{t_i>\tau} \\ U_{t_i}^{\pi} &= \big(Y_{t_i}^{1,\pi}(t_i) - Y_{t_i}^{0,\pi}\big)\mathbb{1}_{t_i\leq\tau} \end{aligned}$$

Now it is easy to give an error estimate of this approximation scheme:

Theorem 5.1. *Under* ***(HF)***, ***(HBQ)***, ***(HFQD)*** *and* ***(HBQD)***, *there exists a constant K such that the error estimate of the approximation scheme is upper bounded by*

$$\sup_{0\leq t\leq T}\mathbb{E}\big[\big|Y_t - Y_{\pi(t)}^{\pi}\big|^2\big] + \mathbb{E}\Big[\int_0^T \big|Z_t - Z_{\pi(t)}^{\pi}\big|^2 dt\Big] + \mathbb{E}\Big[\int_0^T \lambda_t\big|U_t - U_{\pi(t)}^{\pi}\big|^2 dt\Big]$$
$$\leq K\Big(\frac{1}{n^{\frac{2\alpha}{(2-\alpha)(2+K'')-2-2\alpha}}} + \frac{1}{n^{1-K'a+\eta}} + \frac{1}{n}\Big) .$$

Proof. Step 1. *Error for the variable Y.* Fix $t \in [0,T]$. From Theorem 2.1 and (3.13), we have

$$\mathbb{E}\Big[\big|Y_t - Y_t^{\pi}\big|^2\Big] \quad = \quad \mathbb{E}\Big[\big|Y_t^0 - Y_{\pi(t)}^{0,\pi}\big|^2\mathbb{1}_{t<\tau}\Big] + \mathbb{E}\Big[\big|Y_t^1(\tau) - Y_{\pi(t)}^{1,\pi}(\pi(\tau))\big|^2\mathbb{1}_{t\geq\tau}\Big] .$$

Using **(DH)**, we get

$$\begin{aligned} \mathbb{E}\Big[\big|Y_t - Y_t^{\pi}\big|^2\Big] &\leq \mathbb{E}\Big[\big|Y_t^0 - Y_{\pi(t)}^{0,\pi}\big|^2\Big] + \int_0^T \mathbb{E}\Big[\big|Y_t^1(\theta) - Y_{\pi(t)}^{1,\pi}(\pi(\theta))\big|^2\mathbb{1}_{t\geq\theta}\gamma_T(\theta)\Big]d\theta \\ &\leq K\Big(\mathbb{E}\Big[\big|Y_t^0 - Y_{\pi(t)}^{0,\pi}\big|^2\Big] + \sup_{\theta\in[0,T]}\sup_{s\in[\theta,T]}\mathbb{E}\Big[\big|Y_s^1(\theta) - Y_{\pi(s)}^{1,\pi}(\pi(\theta))\big|^2\Big]\Big) . \end{aligned}$$

Using Propositions 5.12 and 5.17, and since t is arbitrary chosen in $[0,T]$, we get

$$\sup_{t\in[0,T]}\mathbb{E}\Big[\big|Y_t - Y_t^{\pi}\big|^2\Big] \quad \leq \quad K\Big(\frac{1}{n^{\frac{2\alpha}{(2-\alpha)(2+K'')-2-2\alpha}}} + \frac{1}{n^{1-K'a+\eta}} + \frac{1}{n}\Big) ,$$

for some constant K which does not depend on π.

Step 2. *Error estimate for the variable Z.* From Theorem 2.1 and (3.13), we have

$$\mathbb{E}\Big[\int_0^T \big|Z_t - Z_t^\pi\big|^2 dt\Big] = \mathbb{E}\Big[\int_0^{T\wedge\tau} \big|Z_t^0 - Z_{\pi(t)}^{0,\pi}\big|^2 dt\Big] + \mathbb{E}\Big[\int_{T\wedge\tau}^T \big|Z_t^1(\tau) - Z_{\pi(t)}^{1,\pi}(\pi(\tau))\big|^2 dt\Big] .$$

Using **(DH)**, we get

$$\begin{aligned}\mathbb{E}\Big[\int_0^T \big|Z_t - Z_t^\pi\big|^2 dt\Big] &= \int_0^T \int_0^\theta \mathbb{E}\Big[\big|Z_t^0 - Z_{\pi(t)}^{0,\pi}\big|^2 \gamma_T(\theta)\Big] dt d\theta \\ &\quad + \int_0^T \int_\theta^T \mathbb{E}\Big[\big|Z_t^1(\theta) - Z_{\pi(t)}^{1,\pi}(\pi(\theta))\big|^2 \gamma_T(\theta)\Big] dt d\theta . \\ &\leq K\Big(\mathbb{E}\Big[\int_0^T \big|Z_t^0 - Z_{\pi(t)}^{0,\pi}\big|^2 dt\Big] + \sup_{\theta\in[0,T]} \mathbb{E}\Big[\int_\theta^T \big|Z_t^1(\theta) - Z_{\pi(t)}^{1,\pi}(\pi(\theta))\big|^2\Big] dt\Big) .\end{aligned}$$

From Propositions 5.12 and 5.17, we get

$$\mathbb{E}\Big[\int_0^T \big|Z_t - Z_t^\pi\big|^2 dt\Big] \leq K\Big(\frac{1}{n^{\frac{2\alpha}{(2-\alpha)(2+K'')-2-2\alpha}}} + \frac{1}{n^{1-K'a+\eta}} + \frac{1}{n}\Big) ,$$

for some constant K which does not depend on π.

Step 3. *Error estimate for the variable U.* From Theorem 2.1 and (3.13), we have

$$\mathbb{E}\Big[\int_0^T \big|U_t - U_t^\pi\big|^2 \lambda_t dt\Big] \leq K\,\mathbb{E}\Big[\int_0^T \Big(|Y_t^1(t) - Y_{\pi(t)}^{1,\pi}(\pi(t))|^2 + |Y_t^0 - Y_{\pi(t)}^{0,\pi}|^2\Big)\lambda_t dt\Big] .$$

Using **(HBI)**, we get

$$\mathbb{E}\Big[\int_0^T \big|U_t - U_t^\pi\big|^2 \lambda_t dt\Big] \leq K\Big(\sup_{\theta\in[0,T]}\sup_{t\in[\theta,T]} \mathbb{E}\Big[\big|Y_t^1(\theta) - Y_{\pi(t)}^{1,\pi}(\pi(\theta))\big|^2\Big] + \sup_{t\in[0,T]} \mathbb{E}\Big[\big|Y_t^0 - Y_{\pi(t)}^{0,\pi}\big|^2\Big]\Big) .$$

Combining this last inequality with Propositions 5.12 and 5.17, we get the result.

□

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