

A decomposition approach for the discrete-time approximation of FBSDEs with a jump I: the Lipschitz case

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Abstract

We study the discrete-time approximation for solutions of forward-backward stochastic differential equations (FBSDEs) with a jump. In this part, we study the case of Lipschitz generators, and we refer to the second part of this work [11] for the quadratic case. Our method is based on a result given in the companion paper [10] which allows to link a FBSDE with a jump with a recursive system of two Brownian FBSDEs. Then we use the classical results on discretization of Brownian FBSDEs to approximate each Brownian FBSDE of recursive system and we recombine these approximations to get a discretization of the FBSDE with a jump. This approach allows to get a convergence rate similar to that of schemes for Brownian FBSDEs.

Keywords: discrete-time approximation, forward-backward SDE, Lipschitz generator, progressive enlargement of filtrations, decomposition in the reference filtration.

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1 Introduction

In this paper, we study a discrete-time approximation for the solution of a forward-backward stochastic differential equation (FBSDE) with a jump of the form

$$\begin{cases} X_t = x + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s + \int_0^t \beta(s, X_{s-})dH_s, \\ Y_t = g(X_T) + \int_t^T f(s, X_s, Y_s, Z_s, U_s)ds - \int_t^T Z_s dW_s - \int_t^T U_s dH_s, \end{cases}$$

where $H_t = \mathbf{1}_{\tau \leq t}$ and τ is a jump time, which can represent a default time in credit risk or counterparty risk. Such equations naturally appear in finance, see for example Bielecki and Jeanblanc [2], Lim and Quenez [14], Peng and Xu [15], Ankirchner *et al.* [1] for an application to exponential utility maximization problem and Kharroubi and Lim [10] for the hedging problem in a complete market. Our work is divided into two parts. In this part, we study the case where the generator f is Lipschitz. The case of a generator f with quadratic growth w.r.t. Z is studied in the second part [11].

For Lipschitz generators, the discrete-time approximation of FBSDEs with jumps is studied by Bouchard and Elie [4] in the case of Poissonian jumps. Their approach is based on a regularity result for the process Z , which is given by Malliavin calculus tools. This regularity result for the process Z was first proved by Zhang [16] in a Brownian framework to provide a convergence rate for the discrete-time approximation of FBSDEs. The use of Malliavin calculus to prove regularity on Z is possible in [4] since the authors suppose that the Brownian motion is independent of the jump measure.

In our case, we only assume that the random jump time τ admits a conditional density given W , which is assumed to be absolutely continuous w.r.t. the Lebesgue measure. In particular, we do not specify a particular law for τ and we do not assume that τ is independent of W as for the case of a Poisson random measure.

To the best of our knowledge, no Malliavin calculus theory has been set for such a framework. Thus, the method used in [4] fails to provide a convergence rate for the approximation in this context.

We therefore follow another approach, which consists in using the decomposition result given in the companion paper [10] to write the solution of a FBSDE with a jump as a combination of solutions to a recursive system of FBSDEs without jump. We then prove a regularity result on the Z components of Brownian BSDEs coming from the decomposition of the BSDE with a jump. This regularity result allows to get a rate for the convergence of the discrete-time schemes for these BSDEs as in [16] or [4].

Finally, we recombine the approximations of the solutions to recursive system of Brownian FBSDEs to get a discretization of the solution to the FBSDE with a jump.

We notice that our approach also allows to weaken the assumption on the forward jump coefficient. More precisely, we only assume that β is Lipschitz continuous, contrary to [4] who assumes that β is regular and the matrix $I_d + \nabla\beta$ is elliptic.

As said upper, this kind of FBSDEs with a jump appears in finance. The wide assumptions made on the jump time τ allow to modelize general phenomenon as a firm default or simpler as a jump of an asset that can be seen as contagion from the default of another firm on the market, see e.g. [9] for some examples. In particular, the approximation of these FBSDEs has its own interest, since it provides approximations of optimal gains and strategies of the studied investment problems. We study in this part the case of FBSDEs with Lipschitz generators, which is related to valuation in complete markets (see [10]) and the utility maximization in incomplete markets with compact investment constraints (see [14]). The study of the discretization of FBSDEs with a quadratic generator, which are related to more general investment problems (see [1]), is postponed to the second part of this work.

We choose to present our results in the case of a single jump and a one-dimensional Brownian motion for the sake of simplicity. We notice that they can easily be extended to the case of a d -dimensional Brownian motion and multiple jumps with eventually random marks, as in [10], taking values in a finite space.

The paper is organized as follows. The next section presents the framework of progressive enlargement of a Brownian filtration by a random jump, and the well posedness of FBSDEs in this context. In Section 3, we present the discrete-time schemes for the forward and backward solutions based on the decomposition given in the previous section. Finally, in Sections 4 and 5, we study the convergence rate of these schemes respectively for the forward and the backward solutions.

2 Preliminaries

2.1 Notation

Throughout this paper, we let $(\Omega, \mathcal{G}, \mathbb{P})$ a probability space on which is defined a standard one dimensional Brownian motion W . We denote $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ the natural filtration of W augmented by all the \mathbb{P} -null sets. We also consider on this space a random time τ , i.e. a nonnegative \mathcal{G} -measurable random variable, and we denote classically the associated jump process by H which is given by

$$H_t = \mathbf{1}_{\tau \leq t}, \quad t \geq 0.$$

We denote by $\mathbb{D} = (\mathcal{D}_t)_{t \geq 0}$ the smallest right-continuous filtration for which τ is a stopping time. The global information is then defined by the progressive enlargement $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ of the initial filtration where $\mathbb{G} := \mathbb{F} \vee \mathbb{D}$. This kind of enlargement was introduced by Jacod, Jeulin and Yor in the 80s (see e.g. [6], [7] and [5]). We introduce some notations used throughout the paper:

- $\mathcal{P}(\mathbb{F})$ (resp. $\mathcal{P}(\mathbb{G})$) is the σ -algebra of \mathbb{F} (resp. \mathbb{G})-predictable measurable subsets of $\Omega \times \mathbb{R}_+$, i.e. the σ -algebra generated by the left-continuous \mathbb{F} (resp. \mathbb{G})-adapted processes,

- $\mathcal{PM}(\mathbb{F})$ (resp. $\mathcal{PM}(\mathbb{G})$) is the σ -algebra of \mathbb{F} (resp. \mathbb{G})-progressively measurable subsets of $\Omega \times \mathbb{R}_+$.

We shall make, throughout the sequel, the standing assumption in the progressive enlargement of filtrations known as density assumption (see e.g. [8, 9, 10]).

(DH) There exists a positive and bounded $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+)$ -measurable process γ such that

$$\mathbb{P}[\tau \in d\theta \mid \mathcal{F}_t] = \gamma_t(\theta)d\theta, \quad t \geq 0.$$

Using Proposition 2.1 in [10] we get that **(DH)** ensures that the process H admits an intensity.

Proposition 2.1. *The process H admits a compensator of the form $\lambda_t dt$, where the process λ is defined by*

$$\lambda_t = \frac{\gamma_t(t)}{\mathbb{P}[\tau > t \mid \mathcal{F}_t]} \mathbf{1}_{t \leq \tau}, \quad t \geq 0.$$

We impose the following assumption to the process λ :

(HBI) The process λ is bounded.

We also introduce the martingale invariance assumption known as the **(H)**-hypothesis.

(H) Any \mathbb{F} -martingale remains a \mathbb{G} -martingale.

We now introduce the following spaces, where $a, b \in \mathbb{R}_+$ with $a \leq b$, and $T < \infty$ is the terminal time:

- $\mathcal{S}_{\mathbb{G}}^{\infty}[a, b]$ (resp. $\mathcal{S}_{\mathbb{F}}^{\infty}[a, b]$) is the set of $\mathcal{PM}(\mathbb{G})$ (resp. $\mathcal{PM}(\mathbb{F})$)-measurable processes $(Y_t)_{t \in [a, b]}$ essentially bounded:

$$\|Y\|_{\mathcal{S}^{\infty}[a, b]} := \operatorname{ess\,sup}_{t \in [a, b]} |Y_t| < \infty.$$

- $\mathcal{S}_{\mathbb{G}}^p[a, b]$ (resp. $\mathcal{S}_{\mathbb{F}}^p[a, b]$), with $p \geq 2$, is the set of $\mathcal{PM}(\mathbb{G})$ (resp. $\mathcal{PM}(\mathbb{F})$)-measurable processes $(Y_t)_{t \in [a, b]}$ such that

$$\|Y\|_{\mathcal{S}^p[a, b]} := \left(\mathbb{E} \left[\sup_{t \in [a, b]} |Y_t|^p \right] \right)^{\frac{1}{p}} < \infty.$$

- $H_{\mathbb{G}}^p[a, b]$ (resp. $H_{\mathbb{F}}^p[a, b]$), with $p \geq 2$, is the set of $\mathcal{P}(\mathbb{G})$ (resp. $\mathcal{P}(\mathbb{F})$)-measurable processes $(Z_t)_{t \in [a, b]}$ such that

$$\|Z\|_{H^p[a, b]} := \mathbb{E} \left[\left(\int_a^b |Z_t|^2 dt \right)^{\frac{p}{2}} \right]^{\frac{1}{p}} < \infty.$$

- $L^2(\lambda)$ is the set of $\mathcal{P}(\mathbb{G})$ -measurable processes $(U_t)_{t \in [0, T]}$ such that

$$\|U\|_{L^2(\mu)} := \left(\mathbb{E} \left[\int_0^T |U_s|^2 \lambda_s ds \right] \right)^{\frac{1}{2}} < \infty.$$

2.2 Forward-Backward SDE with a jump

Given measurable functions $b : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$, $\sigma : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$, $\beta : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$, $g : \mathbb{R} \rightarrow \mathbb{R}$ and $f : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, and an initial condition $x \in \mathbb{R}$, we study the discrete-time approximation of the solution (X, Y, Z, U) in $\mathcal{S}_{\mathbb{C}}^2[0, T] \times \mathcal{S}_{\mathbb{C}}^\infty[0, T] \times H_{\mathbb{C}}^2[0, T] \times L^2(\lambda)$ to the following forward-backward stochastic differential equation:

$$X_t = x + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dW_s + \int_0^t \beta(s, X_{s-}) dH_s, \quad 0 \leq t \leq T, \quad (2.1)$$

$$Y_t = g(X_T) + \int_t^T f(s, X_s, Y_s, Z_s, U_s(1 - H_s)) ds - \int_t^T Z_s dW_s - \int_t^T U_s dH_s, \quad 0 \leq t \leq T, \quad (2.2)$$

when the generator of the BSDE is Lipschitz.

Remark 2.1. In BSDE (2.2), the jump component U of the unknown (Y, Z, U) appears in the generator f with an additional multiplicative term $1 - H$. This ensures the equation to be well posed in $\mathcal{S}_{\mathbb{C}}^\infty[0, T] \times H_{\mathbb{C}}^2[0, T] \times L^2(\lambda)$. Indeed, the component U lives in $L^2(\lambda)$, thus its value on $(\tau \wedge T, T]$ is not defined, since the intensity λ vanishes on $(\tau \wedge T, T]$. We therefore introduce the term $1 - H$ to kill the value of U on $(\tau \wedge T, T]$ and hence to avoid making the equation depends on it.

We first prove that the decoupled system (2.1)-(2.2) admits a solution. To this end, we introduce several assumptions on the coefficients b, σ, β, g and f . We consider the following assumption for the forward coefficients:

(HF) There exists a constant K such that the functions b, σ and β satisfy

$$|b(t, 0)| + |\sigma(t, 0)| + |\beta(t, 0)| \leq K,$$

and

$$|b(t, x) - b(t, x')| + |\sigma(t, x) - \sigma(t, x')| + |\beta(t, x) - \beta(t, x')| \leq K|x - x'|,$$

for all $(t, x, x') \in [0, T] \times \mathbb{R} \times \mathbb{R}$.

For the backward coefficients g and f , we impose the following assumption:

(HBL) There exists a constant K such that the functions g and f satisfy

$$|f(t, x, 0, 0, 0)| + |g(x)| \leq K,$$

and

$$|f(t, x, y, z, u) - f(t, x, y', z', u')| \leq K(|y - y'| + |z - z'| + |u - u'|),$$

for all $(t, x, y, y', z, z', u, u') \in [0, T] \times \mathbb{R} \times [\mathbb{R}]^2 \times [\mathbb{R}]^2 \times [\mathbb{R}]^2$.

Following the decomposition approach initiated by [10], we introduce the recursive system of FBSDEs associated with (2.1)-(2.2):

- Find $(X^1(\theta), Y^1(\theta), Z^1(\theta)) \in \mathcal{S}_{\mathbb{F}}^2[0, T] \times \mathcal{S}_{\mathbb{F}}^\infty[\theta, T] \times H_{\mathbb{F}}^2[\theta, T]$ such that

$$X_t^1(\theta) = x + \int_0^t b(s, X_s^1(\theta)) ds + \int_0^t \sigma(s, X_s^1(\theta)) dW_s + \beta(\theta, X_{\theta-}^1(\theta)) \mathbf{1}_{\theta \leq t}, \quad 0 \leq t \leq T, \quad (2.3)$$

$$Y_t^1(\theta) = g(X_T^1(\theta)) + \int_t^T f(s, X_s^1(\theta), Y_s^1(\theta), Z_s^1(\theta), 0) ds - \int_t^T Z_s^1(\theta) dW_s, \quad \theta \leq t \leq T, \quad (2.4)$$

for all $\theta \in [0, T]$.

- Find $(X^0, Y^0, Z^0) \in \mathcal{S}_{\mathbb{F}}^2[0, T] \times \mathcal{S}_{\mathbb{F}}^\infty[0, T] \times H_{\mathbb{F}}^2[0, T]$ such that

$$X_t^0 = x + \int_0^t b(s, X_s^0) ds + \int_0^t \sigma(s, X_s^0) dW_s, \quad 0 \leq t \leq T, \quad (2.5)$$

$$Y_t^0 = g(X_T^0) + \int_t^T f(s, X_s^0, Y_s^0, Z_s^0, Y_s^1(s) - Y_s^0) ds - \int_t^T Z_s^0 dW_s, \quad 0 \leq t \leq T. \quad (2.6)$$

Then, the link between FBSDE (2.1)-(2.2) and the recursive system of FBSDEs (2.5)-(2.6) and (2.3)-(2.4) is given by the following result.

Theorem 2.1. *Assume that **(HF)** and **(HBL)** hold true. Then, FBSDE (2.1)-(2.2) admits a unique solution $(X, Y, Z, U) \in \mathcal{S}_{\mathbb{G}}^2[0, T] \times \mathcal{S}_{\mathbb{G}}^\infty[0, T] \times H_{\mathbb{G}}^2[0, T] \times L^2(\lambda)$ given by*

$$\begin{cases} X_t = X_t^0 \mathbf{1}_{t < \tau} + X_t^1(\tau) \mathbf{1}_{\tau \leq t}, \\ Y_t = Y_t^0 \mathbf{1}_{t < \tau} + Y_t^1(\tau) \mathbf{1}_{\tau \leq t}, \\ Z_t = Z_t^0 \mathbf{1}_{t \leq \tau} + Z_t^1(\tau) \mathbf{1}_{\tau < t}, \\ U_t = (Y_t^1(t) - Y_t^0) \mathbf{1}_{t \leq \tau}, \end{cases} \quad (2.7)$$

where $(X^1(\theta), Y^1(\theta), Z^1(\theta))$ is the unique solution to FBSDE (2.3)-(2.4) in $\mathcal{S}_{\mathbb{F}}^2[0, T] \times \mathcal{S}_{\mathbb{F}}^\infty[\theta, T] \times H_{\mathbb{F}}^2[\theta, T]$, for $\theta \in [0, T]$, and (X^0, Y^0, Z^0) is the unique solution to FBSDE (2.5)-(2.6) in $\mathcal{S}_{\mathbb{F}}^2[0, T] \times \mathcal{S}_{\mathbb{F}}^\infty[0, T] \times H_{\mathbb{F}}^2[0, T]$.

Proof.

Step 1. Solution to (2.1) under **(HF)**.

Under **(HF)** there exist unique processes $X^0 \in \mathcal{S}_{\mathbb{F}}^2[0, T]$ satisfying (2.5), and $X^1(\theta) \in \mathcal{S}_{\mathbb{F}}^2[0, T]$ satisfying (2.3) for all $\theta \in [0, T]$ such that X^1 is $\mathcal{P}\mathcal{M}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+)$ -measurable. Then, from the definition of H , we check that the process X defined by

$$X_t = X_t^0 \mathbf{1}_{t < \tau} + X_t^1(\tau) \mathbf{1}_{t \geq \tau}, \quad (2.8)$$

satisfies (2.1). We now check that $X \in \mathcal{S}_{\mathbb{G}}^2[0, T]$. We first notice that from **(HF)**, there exists a constant K such that

$$\mathbb{E} \left[\sup_{t \in [0, T]} |X_t^0|^2 \right] \leq K. \quad (2.9)$$

Then, from the definition of X^1 , we have for all $t \in [\theta, T]$

$$\sup_{s \in [\theta, t]} |X_s^1(\theta)|^2 \leq K \left(|X_\theta^0|^2 + |\beta(\theta, X_\theta^0)|^2 + \int_\theta^t |b(u, X_u^1(\theta))|^2 du + \sup_{s \in [\theta, t]} \left| \int_\theta^s \sigma(u, X_u^1(\theta)) dW_u \right|^2 \right).$$

Using **(HF)** and BDG-inequality, we get

$$\mathbb{E} \left[\sup_{s \in [\theta, t]} |X_s^1(\theta)|^2 \right] \leq K \left(1 + \int_\theta^t \mathbb{E} \left[\sup_{u \in [\theta, s]} |X_u^1(\theta)|^2 \right] du \right),$$

for some constant K which does not depend on θ . Applying Gronwall's lemma, we get

$$\sup_{\theta \in [0, T]} \|X^1(\theta)\|_{\mathcal{S}_{\mathbb{F}}^2[\theta, T]} \leq K. \quad (2.10)$$

Combining (2.8), (2.9) and (2.10), we get that $X \in \mathcal{S}_{\mathbb{G}}^2[0, T]$. Moreover still using **(HF)** we get the uniqueness of a solution to (2.1) in $\mathcal{S}_{\mathbb{G}}^2[0, T]$.

Step 2. Solution to (2.2) under **(HBL)**.

To follow the decomposition approach initiated by the authors in [10], we need the generator to be predictable. To this end, we notice that in BSDE (2.2), we can replace the generator $(t, y, z, u) \mapsto f(t, X_t, y, z, u(1 - H_t))$ by the predictable map $(t, y, z, u) \mapsto f(t, X_{t-}, y, z, u(1 - H_{t-}))$.

Using the decomposition (2.8), we are able to write explicitly the decompositions of the \mathcal{G}_T -measurable random variable $g(X_T)$ and the $\mathcal{P}(\mathbb{G}) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$ -measurable map $(\omega, t, y, z, u) \mapsto f(t, X_{t-}(\omega), y, z, u(1 - H_{t-}(\omega)))$ given by Lemma 2.1 in [10]:

$$\begin{aligned} g(X_T) &= g(X_T^0) \mathbf{1}_{T < \tau} + g(X_T^1(\tau)) \mathbf{1}_{T \geq \tau}, \\ f(t, X_{t-}, y, z, (1 - H_{t-})u) &= f^0(t, y, z, u) \mathbf{1}_{t \leq \tau} + f^1(t, y, z, u, \tau) \mathbf{1}_{t > \tau}, \end{aligned}$$

with $f^0(t, y, z, u) = f(t, X_t^0, y, z, u)$ and $f^1(t, y, z, u, \tau) = f(t, X_{t-}^1(\tau), y, z, 0)$, for all $(t, y, z, u, \theta) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+$.

Suppose now that **(HBL)** holds true. Then, from Proposition C.1 in [10], BSDE (2.4) admits a $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}([0, T])$ -measurable solution (Y^1, Z^1) and BSDE (2.6) admits a solution (Y^0, Z^0) . Using Proposition 2.1 in [13], we obtain the existence of a constant K such that

$$\|Y^1(\theta)\|_{\mathcal{S}^\infty[\theta, T]} + \|Z^1(\theta)\|_{H^2[\theta, T]} \leq K,$$

for all $\theta \in [0, T]$, and

$$\|Y^0\|_{\mathcal{S}^\infty[0, T]} + \|Z^0\|_{H^2[0, T]} \leq K.$$

We can then apply Theorem 3.1 in [10] and we get the existence of a solution to (2.2) in $\mathcal{S}_{\mathbb{G}}^\infty[0, T] \times H_{\mathbb{G}}^2[0, T] \times L^2(\lambda)$.

Let (Y, Z, U) and (Y', Z', U') be two solutions to (2.2) in $\mathcal{S}_{\mathbb{G}}^\infty[0, T] \times H_{\mathbb{G}}^2[0, T] \times L^2(\lambda)$. Since $f(t, x, y, z, u(1 - H_t)) = f(t, x, y, z, 0)$ for all $t \in (\tau \wedge T, T]$ and λ vanishes on $(\tau \wedge T, T]$,

we can assume w.l.o.g. that $U_t = U'_t = 0$ for $t \in (\tau \wedge T, T]$. Then, from **(HBL)**, we can apply Theorem 4.1 in [10] and we get that $Y \leq Y'$. Since Y and Y' play the same role, we obtain $Y = Y'$. Identifying the pure jump parts of Y and Y' gives $U = U'$. Finally, identifying the unbounded variation gives $Z = Z'$. \square

Throughout the sequel, we give an approximation of the solution to FBSDE (2.1)-(2.2) by studying the approximation of the solutions to the recursive system of FBSDEs (2.3)-(2.4) and (2.5)-(2.6). For that we use the classical results of discretization in the case of Lipschitz Brownian FBSDEs.

3 Discrete-time scheme for the FBSDE

In this section, we introduce a discrete-time approximation of the solution (X, Y, Z, U) to FBSDE (2.1)-(2.2) based on its decomposition given by Theorem 2.1.

Throughout the sequel, we consider a discretization grid $\pi = \{t_0, \dots, t_n\}$ of $[0, T]$ with $0 = t_0 < t_1 < \dots < t_n = T$. For $t \in [0, T]$, we denote by $\pi(t)$ the largest element of π smaller than t :

$$\pi(t) := \max \{ t_i, i = 0, \dots, n \mid t_i \leq t \} .$$

We also denote by $|\pi|$ the mesh of π :

$$|\pi| := \max \{ t_{i+1} - t_i, i = 0, \dots, n-1 \} ,$$

that we suppose satisfying $|\pi| \leq 1$, and by ΔW_i^π (resp. Δt_i^π) the increment of W (resp. the difference) between t_i and t_{i-1} : $\Delta W_i^\pi := W_{t_i} - W_{t_{i-1}}$ (resp. $\Delta t_i^\pi := t_i - t_{i-1}$), for $1 \leq i \leq n$.

3.1 Discrete-time scheme for X

We introduce an approximation of the process X based on the discretization of the processes X^0 and X^1 .

- *Euler scheme for X^0* . We consider the classical scheme $X^{0,\pi}$ defined by

$$\begin{cases} X_{t_0}^{0,\pi} = x , \\ X_{t_i}^{0,\pi} = X_{t_{i-1}}^{0,\pi} + b(t_{i-1}, X_{t_{i-1}}^{0,\pi})\Delta t_i^\pi + \sigma(t_{i-1}, X_{t_{i-1}}^{0,\pi})\Delta W_i^\pi , \quad 1 \leq i \leq n . \end{cases} \quad (3.1)$$

- *Euler scheme for X^1* . Since the process X^1 depends on two parameters t and θ , we introduce a discretization of X^1 in these two variables. We then consider the following scheme

$$\begin{cases} X_{t_0}^{1,\pi}(\pi(\theta)) = x + \beta(t_0, x)\mathbb{1}_{\pi(\theta)=0} , \\ X_{t_i}^{1,\pi}(\pi(\theta)) = X_{t_{i-1}}^{1,\pi}(\pi(\theta)) + b(t_{i-1}, X_{t_{i-1}}^{1,\pi}(\pi(\theta)))\Delta t_i^\pi + \sigma(t_{i-1}, X_{t_{i-1}}^{1,\pi}(\pi(\theta)))\Delta W_i^\pi \\ \quad + \beta(t_{i-1}, X_{t_{i-1}}^{1,\pi}(\pi(\theta)))\mathbb{1}_{t_i=\pi(\theta)} , \quad 1 \leq i \leq n , \quad 0 \leq \theta \leq T . \end{cases} \quad (3.2)$$

We are now able to provide an approximation of the process X solution to FSDE (2.1). We consider the scheme X^π defined by

$$X_t^\pi = X_{\pi(t)}^{0,\pi} \mathbf{1}_{t < \tau} + X_{\pi(t)}^{1,\pi}(\pi(\tau)) \mathbf{1}_{t \geq \tau}, \quad 0 \leq t \leq T. \quad (3.3)$$

We shall denote by $\{\mathcal{F}_i^{0,\pi}\}_{0 \leq i \leq n}$ (resp. $\{\mathcal{F}_i^{1,\pi}(\theta)\}_{0 \leq i \leq n}$) the discrete-time filtration associated with $X^{0,\pi}$ (resp. $X^{1,\pi}$)

$$\begin{aligned} \mathcal{F}_i^{0,\pi} &:= \sigma(X_{t_j}^{0,\pi}, j \leq i) \\ (\text{resp. } \mathcal{F}_i^{1,\pi}(\theta) &:= \sigma(X_{t_j}^{1,\pi}(\theta), j \leq i)). \end{aligned}$$

3.2 Discrete-time scheme for (Y, Z, U)

We introduce an approximation of (Y, Z) based on the discretization of (Y^0, Z^0) and (Y^1, Z^1) . To this end we introduce the backward implicit schemes on π associated with BSDEs (2.4) and (2.6). Since the system is recursively coupled, we first introduce the scheme associated with (2.4). We then use it to define the scheme associated with (2.6).

• *Backward Euler scheme for (Y^1, Z^1) .* We consider the classical implicit scheme $(Y^{1,\pi}, Z^{1,\pi})$ defined by

$$\begin{cases} Y_T^{1,\pi}(\pi(\theta)) = g(X_T^{1,\pi}(\pi(\theta))), \\ Y_{t_{i-1}}^{1,\pi}(\pi(\theta)) = \mathbb{E}_{i-1}^{1,\pi(\theta)}[Y_{t_i}^{1,\pi}(\pi(\theta))] + f(t_{i-1}, X_{t_{i-1}}^{1,\pi}(\pi(\theta)), Y_{t_{i-1}}^{1,\pi}(\pi(\theta)), Z_{t_{i-1}}^{1,\pi}(\pi(\theta)), 0) \Delta t_i^\pi, \\ Z_{t_{i-1}}^{1,\pi}(\pi(\theta)) = \frac{1}{\Delta t_i^\pi} \mathbb{E}_{i-1}^{1,\pi(\theta)}[Y_{t_i}^{1,\pi}(\pi(\theta)) \Delta W_i^\pi], \quad \pi(\theta) \leq t_{i-1}, \quad 1 \leq i \leq n, \end{cases} \quad (3.4)$$

where $\mathbb{E}_i^{1,\theta} = \mathbb{E}[\cdot | \mathcal{F}_i^{1,\pi}(\theta)]$ for $0 \leq i \leq n$ and $\theta \in [0, T]$.

• *Backward Euler scheme for (Y^0, Z^0) .* Since the generator of (2.6) involves the process $(Y_t^1(t))_{t \in [0, T]}$, we consider a discretization based on $Y^{1,\pi}$. We therefore consider the scheme $(Y^{0,\pi}, Z^{0,\pi})$ defined by

$$\begin{cases} Y_T^{0,\pi} = g(X_T^{0,\pi}), \\ Y_{t_{i-1}}^{0,\pi} = \mathbb{E}_{i-1}^0[Y_{t_i}^{0,\pi}] + \bar{f}^\pi(t_{i-1}, X_{t_{i-1}}^{0,\pi}, Y_{t_{i-1}}^{0,\pi}, Z_{t_{i-1}}^{0,\pi}) \Delta t_i^\pi, \\ Z_{t_{i-1}}^{0,\pi} = \frac{1}{\Delta t_i^\pi} \mathbb{E}_{i-1}^0[Y_{t_i}^{0,\pi} \Delta W_i^\pi], \quad 1 \leq i \leq n, \end{cases} \quad (3.5)$$

where $\mathbb{E}_i^0 = \mathbb{E}[\cdot | \mathcal{F}_i^{0,\pi}]$ for $0 \leq i \leq n$, and \bar{f}^π is defined by

$$\bar{f}^\pi(t, x, y, z) = f(t, x, y, z, Y_{\pi(t)}^{1,\pi}(\pi(t)) - y),$$

for all $(t, x, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$.

We then consider the following scheme for the solution (Y, Z, U) of BSDE (2.2)

$$\begin{cases} Y_t^\pi = Y_{\pi(t)}^{0,\pi} \mathbf{1}_{t < \tau} + Y_{\pi(t)}^{1,\pi}(\pi(\tau)) \mathbf{1}_{t \geq \tau}, \\ Z_t^\pi = Z_{\pi(t)}^{0,\pi} \mathbf{1}_{t \leq \tau} + Z_{\pi(t)}^{1,\pi}(\pi(\tau)) \mathbf{1}_{t > \tau}, \\ U_t^\pi = (Y_{\pi(t)}^{1,\pi}(\pi(t)) - Y_{\pi(t)}^{0,\pi}) \mathbf{1}_{t \leq \tau}, \end{cases} \quad (3.6)$$

for $t \in [0, T]$.

4 Convergence of the scheme for the FSDE

We introduce the following assumption, which will be used to discretize X .

(HFD) There exists a constant K such that the functions b , σ and β satisfy

$$\begin{aligned} |b(t, x) - b(t', x)| + |\sigma(t, x) - \sigma(t', x)| &\leq K|t - t'|^{\frac{1}{2}}, \\ |\beta(t, x) - \beta(t', x)| + |\sigma(t, x) - \sigma(t', x)| &\leq K|t - t'|, \end{aligned}$$

for all $(t, t', x) \in [0, T] \times [0, T] \times \mathbb{R}$.

We now provide an error estimate of the approximation schemes for X^0 and X^1 . We then use these estimates to control the error between X and X^π .

4.1 Error estimates for X^0 and X^1

Under **(HF)** and **(HFD)**, the upper bound of the error between X^0 and its Euler scheme $X^{0,\pi}$ is well understood, see e.g. [12], and we have

$$\mathbb{E} \left[\sup_{t \in [0, T]} |X_t^0 - X_{\pi(t)}^{0,\pi}|^2 \right] \leq K|\pi|, \quad (4.1)$$

for some constant K which does not depend on π .

The next result provides an upper bound for the error between X^1 and its Euler scheme $X^{1,\pi}$ defined by (3.2).

Theorem 4.1. *Under **(HF)** and **(HFD)**, we have the following estimate*

$$\sup_{\theta \in [0, T]} \mathbb{E} \left[\sup_{t \in [\theta, T]} |X_t^1(\theta) - X_{\pi(t)}^{1,\pi}(\pi(\theta))|^2 \right] \leq K|\pi|,$$

for a constant K which does not depend on π .

Proof. Fix $\theta \in [0, T]$, we then have

$$\begin{aligned} \mathbb{E} \left[\sup_{t \in [\theta, T]} |X_t^1(\theta) - X_{\pi(t)}^{1,\pi}(\pi(\theta))|^2 \right] &\leq 2 \mathbb{E} \left[\sup_{t \in [\theta, T]} |X_t^1(\theta) - X_t^1(\pi(\theta))|^2 \right] \\ &\quad + 2 \mathbb{E} \left[\sup_{t \in [\theta, T]} |X_t^1(\pi(\theta)) - X_{\pi(t)}^{1,\pi}(\pi(\theta))|^2 \right]. \end{aligned} \quad (4.2)$$

We study separately the two terms of the right hand side.

Since $\pi(\theta) \leq \theta \leq t$, we have by definition $X_s^1(\pi(\theta)) = X_s^0$ for all $s \in [0, \pi(\theta))$, and $X_s^1(\theta) = X_s^0$ for all $s \in [0, \theta)$, which imply

$$\begin{aligned}
X_t^1(\theta) - X_t^1(\pi(\theta)) &= \int_{\pi(\theta)}^{\theta} b(s, X_s^0) ds + \int_{\pi(\theta)}^{\theta} \sigma(s, X_s^0) dW_s + \beta(\theta, X_{\theta}^0) + \int_{\theta}^t b(s, X_s^1(\theta)) ds \\
&+ \int_{\theta}^t \sigma(s, X_s^1(\theta)) dW_s - \beta(\pi(\theta), X_{\pi(\theta)}^0) - \int_{\pi(\theta)}^{\theta} b(s, X_s^1(\pi(\theta))) ds \\
&- \int_{\pi(\theta)}^{\theta} \sigma(s, X_s^1(\pi(\theta))) dW_s - \int_{\theta}^t b(s, X_s^1(\pi(\theta))) ds \\
&- \int_{\theta}^t \sigma(s, X_s^1(\pi(\theta))) dW_s,
\end{aligned}$$

for all $t \in [\theta, T]$.

Hence, there exists a constant K such that

$$\begin{aligned}
|X_t^1(\theta) - X_t^1(\pi(\theta))|^2 &\leq K \left\{ \left| \int_{\pi(\theta)}^{\theta} b(s, X_s^0) ds \right|^2 + \left| \int_{\pi(\theta)}^{\theta} b(s, X_s^1(\pi(\theta))) ds \right|^2 \right. \\
&+ \int_{\theta}^t \left| b(s, X_s^1(\theta)) - b(s, X_s^1(\pi(\theta))) \right|^2 ds \\
&+ \left| \int_{\pi(\theta)}^{\theta} \sigma(s, X_s^0) dW_s \right|^2 + \left| \int_{\pi(\theta)}^{\theta} \sigma(s, X_s^1(\pi(\theta))) dW_s \right|^2 \\
&+ \left| \int_{\theta}^t \left(\sigma(s, X_s^1(\theta)) - \sigma(s, X_s^1(\pi(\theta))) \right) dW_s \right|^2 \\
&\left. + \left| \beta(\theta, X_{\theta}^0) - \beta(\pi(\theta), X_{\pi(\theta)}^0) \right|^2 \right\}. \tag{4.3}
\end{aligned}$$

From **(HF)** and **(HFD)**, we have

$$\mathbb{E} \left| \beta(\theta, X_{\theta}^0) - \beta(\pi(\theta), X_{\pi(\theta)}^0) \right|^2 \leq K (|\pi|^2 + \mathbb{E} |X_{\theta}^0 - X_{\pi(\theta)}^0|^2).$$

We have from **(HF)**

$$\mathbb{E} \left[\left| \int_{\pi(\theta)}^{\theta} b(s, X_s^0) ds \right|^2 + \left| \int_{\pi(\theta)}^{\theta} \sigma(s, X_s^0) dW_s \right|^2 \right] \leq K |\pi|,$$

which implies in particular $\mathbb{E} |X_{\theta}^0 - X_{\pi(\theta)}^0|^2 \leq K |\pi|$ and hence

$$\mathbb{E} \left| \beta(\theta, X_{\theta}^0) - \beta(\pi(\theta), X_{\pi(\theta)}^0) \right|^2 \leq K |\pi|.$$

We have also from **(HF)** and (2.10)

$$\mathbb{E} \left[\left| \int_{\pi(\theta)}^{\theta} b(s, X_s^1(\pi(\theta))) ds \right|^2 + \left| \int_{\pi(\theta)}^{\theta} \sigma(s, X_s^1(\pi(\theta))) dW_s \right|^2 \right] \leq K |\pi|,$$

where K does not depend on θ .

Combining these inequalities with (4.3), **(HF)** and BDG-inequality, we get

$$\mathbb{E} \left[\sup_{u \in [\theta, t]} |X_u^1(\theta) - X_u^1(\pi(\theta))|^2 \right] \leq K \left(\int_{\theta}^t \mathbb{E} \left[\sup_{u \in [\theta, s]} |X_u^1(\theta) - X_u^1(\pi(\theta))|^2 \right] ds + |\pi| \right).$$

Applying Gronwall's lemma, we get

$$\mathbb{E} \left[\sup_{t \in [\theta, T]} |X_t^1(\theta) - X_t^1(\pi(\theta))|^2 \right] \leq K|\pi|, \quad (4.4)$$

where K does not depend on θ .

To find an upper bound for the term $\mathbb{E}[\sup_{t \in [\theta, T]} |X_t^1(\pi(\theta)) - X_{\pi(t)}^{1, \pi}(\pi(\theta))|^2]$ we introduce the scheme $\tilde{X}^{\pi}(\pi(\theta))$ defined by

$$\begin{cases} \tilde{X}_{\pi(\theta)}^{\pi}(\pi(\theta)) = X_{\pi(\theta)}^1(\pi(\theta)), \\ \tilde{X}_{t_i}^{\pi}(\pi(\theta)) = \tilde{X}_{t_{i-1}}^{\pi}(\pi(\theta)) + b(t_{i-1}, \tilde{X}_{t_{i-1}}^{\pi}(\pi(\theta))) \Delta t_i^{\pi} + \sigma(t_{i-1}, \tilde{X}_{t_{i-1}}^{\pi}(\pi(\theta))) \Delta W_i^{\pi}, \quad t_i > \pi(\theta). \end{cases}$$

We have the inequality

$$\begin{aligned} \mathbb{E} \left[\sup_{t \in [\theta, T]} |X_t^1(\pi(\theta)) - X_{\pi(t)}^{1, \pi}(\pi(\theta))|^2 \right] &\leq 2 \mathbb{E} \left[\sup_{t \in [\theta, T]} |X_t^1(\pi(\theta)) - \tilde{X}_{\pi(t)}^{\pi}(\pi(\theta))|^2 \right] \\ &\quad + 2 \mathbb{E} \left[\sup_{t \in [\theta, T]} |\tilde{X}_{\pi(t)}^{\pi}(\pi(\theta)) - X_{\pi(t)}^{1, \pi}(\pi(\theta))|^2 \right]. \end{aligned} \quad (4.5)$$

Since $\tilde{X}^{\pi}(\pi(\theta))$ is the Euler scheme of $X^1(\pi(\theta))$ on $[\pi(\theta), T]$, we have under **(HF)** and **(HFD)** (see e.g. [12])

$$\mathbb{E} \left[\sup_{t \in [\theta, T]} |X_t^1(\pi(\theta)) - \tilde{X}_{\pi(t)}^{\pi}(\pi(\theta))|^2 \right] \leq K \left(1 + \mathbb{E} \left[|X_{\pi(\theta)}^1(\pi(\theta))|^2 \right] \right) |\pi|,$$

for some constant K which neither depends on π nor on θ . From (2.10), we get

$$\mathbb{E} \left[\sup_{t \in [\theta, T]} |X_t^1(\pi(\theta)) - \tilde{X}_{\pi(t)}^{\pi}(\pi(\theta))|^2 \right] \leq K|\pi|, \quad (4.6)$$

for all $\theta \in [0, T]$.

We now study the term $\mathbb{E}[\sup_{t \in [\theta, T]} |\tilde{X}_{\pi(t)}^{\pi}(\pi(\theta)) - X_{\pi(t)}^{1, \pi}(\pi(\theta))|^2]$. We first notice that we have the following identity

$$\mathbb{E} \left[\sup_{t \in [\theta, T]} |\tilde{X}_{\pi(t)}^{\pi}(\pi(\theta)) - X_{\pi(t)}^{1, \pi}(\pi(\theta))|^2 \right] = \mathbb{E} \left[\sup_{t \in [\pi(\theta), T]} |\tilde{X}_{\pi(t)}^{\pi}(\pi(\theta)) - X_{\pi(t)}^{1, \pi}(\pi(\theta))|^2 \right].$$

Hence we can work with the second term. From the definition of \tilde{X}^{π} and $X^{1, \pi}$, there exists a constant K such that

$$\begin{aligned} &\sup_{u \in [\pi(\theta), t]} |\tilde{X}_{\pi(u)}^{\pi}(\pi(\theta)) - X_{\pi(u)}^{1, \pi}(\pi(\theta))|^2 \leq \\ &K \left(|X_{\pi(\theta)}^1(\pi(\theta)) - X_{\pi(\theta)}^{1, \pi}(\pi(\theta))|^2 + \int_{\pi(\theta)}^{\pi(t)} \left| b(\pi(s), \tilde{X}_{\pi(s)}^{\pi}(\pi(\theta))) - b(\pi(s), X_{\pi(s)}^{1, \pi}(\pi(\theta))) \right|^2 ds \right. \\ &\quad \left. + \sup_{u \in [\pi(\theta), t]} \left| \int_{\pi(\theta)}^{\pi(u)} \left(\sigma(\pi(s), \tilde{X}_{\pi(s)}^{\pi}(\pi(\theta))) - \sigma(\pi(s), X_{\pi(s)}^{1, \pi}(\pi(\theta))) \right) dW_s \right|^2 \right). \end{aligned}$$

Then, using **(HF)** and BDG-inequality, we get

$$\begin{aligned} \mathbb{E} \left[\sup_{u \in [\pi(\theta), t]} |\tilde{X}_{\pi(u)}^\pi(\pi(\theta)) - X_{\pi(u)}^{1,\pi}(\pi(\theta))|^2 \right] &\leq K \left(\mathbb{E} |X_{\pi(\theta)}^1(\pi(\theta)) - X_{\pi(\theta)}^{1,\pi}(\pi(\theta))|^2 \right. \\ &\quad \left. + \int_{\pi(\theta)}^t \mathbb{E} \left[\sup_{u \in [\pi(\theta), s]} |\tilde{X}_{\pi(u)}^\pi(\pi(\theta)) - X_{\pi(u)}^{1,\pi}(\pi(\theta))|^2 \right] ds \right). \end{aligned}$$

From Lipschitz property of β , we have

$$\begin{aligned} \mathbb{E} |X_{\pi(\theta)}^1(\pi(\theta)) - X_{\pi(\theta)}^{1,\pi}(\pi(\theta))|^2 &= \mathbb{E} |X_{\pi(\theta)}^0 + \beta(\pi(\theta), X_{\pi(\theta)}^0) - X_{\pi(\theta)}^{0,\pi} - \beta(\pi(\theta), X_{\pi(\theta)}^{0,\pi})|^2 \\ &\leq K \mathbb{E} |X_{\pi(\theta)}^0 - X_{\pi(\theta)}^{0,\pi}|^2. \end{aligned}$$

This last inequality with (4.1) gives

$$\mathbb{E} |X_{\pi(\theta)}^1(\pi(\theta)) - X_{\pi(\theta)}^{1,\pi}(\pi(\theta))|^2 \leq K|\pi|.$$

Applying Gronwall's lemma, we get

$$\mathbb{E} \left[\sup_{t \in [\pi(\theta), T]} |\tilde{X}_{\pi(t)}^\pi(\pi(\theta)) - X_{\pi(t)}^{1,\pi}(\pi(\theta))|^2 \right] \leq K|\pi|. \quad (4.7)$$

Combining (4.2), (4.4), (4.5), (4.6) and (4.7), we get the result. \square

4.2 Error estimate for the FSDE with a jump

We are now able to provide an estimate of the error approximation of the process X by its scheme X^π defined by (3.3).

Theorem 4.2. *Under **(HF)** and **(HFD)**, we have the following estimate*

$$\mathbb{E} \left[\sup_{t \in [0, T]} |X_t - X_t^\pi|^2 \right] \leq K|\pi|,$$

for a constant K which does not depend on π .

Proof. From the definition of X^π we have

$$\begin{aligned} \mathbb{E} \left[\sup_{t \in [0, T]} |X_t - X_t^\pi|^2 \right] &\leq \mathbb{E} \left[\sup_{t \in [0, \tau]} |X_t^0 - X_{\pi(t)}^{0,\pi}|^2 \right] + \mathbb{E} \left[\sup_{t \in [\tau, T]} |X_t^1(\tau) - X_{\pi(t)}^{1,\pi}(\pi(\tau))|^2 \right] \\ &\leq \mathbb{E} \left[\sup_{t \in [0, T]} |X_t^0 - X_{\pi(t)}^{0,\pi}|^2 \right] + \int_0^T \mathbb{E} \left[\sup_{t \in [\theta, T]} |X_t^1(\theta) - X_{\pi(t)}^{1,\pi}(\pi(\theta))|^2 \gamma_T(\theta) \right] d\theta. \end{aligned}$$

Using (4.1) and **(DH)**, we have

$$\mathbb{E} \left[\sup_{t \in [0, T]} |X_t - X_t^\pi|^2 \right] \leq K \left(|\pi| + \sup_{\theta \in [0, T]} \mathbb{E} \left[\sup_{s \in [\theta, T]} |X_s^1(\theta) - X_{\pi(s)}^{1,\pi}(\pi(\theta))|^2 \right] \right).$$

From Theorem 4.1, we get

$$\mathbb{E} \left[\sup_{t \in [0, T]} |X_t - X_t^\pi|^2 \right] \leq K|\pi|.$$

\square

5 Convergence of the scheme for the BSDE

To provide error estimates for the Euler scheme of the BSDE, we need an additional regularity property for the coefficients g and f . We then introduce the following assumption.

(HBLD) There exists a constant K such that the functions g and f satisfy

$$|g(x) - g(x')| + |f(t, x, y, z, u) - f(t', x', y, z, u)| \leq K(|x - x'| + |t - t'|^{\frac{1}{2}}),$$

for all $(t, t', x, x', y, z, u) \in [0, T]^2 \times [\mathbb{R}]^2 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$.

We are now ready to provide error estimates of the approximation schemes for (Y^0, Z^0) and (Y^1, Z^1) , and then for (Y, Z) .

5.1 Regularity results

In this part, we give some results on the regularity of the processes Z^1 and Z^0 . We denote by $\mathcal{F}_t^0 = \sigma\{X_s^0, 0 \leq s \leq t\}$ and $\mathcal{F}_t^1(\theta) = \sigma\{X_s^1(\theta), \theta \leq s \leq t\}$.

Proposition 5.1. *Under **(HF)**, **(HFD)**, **(HBL)** and **(HBLD)**, there exists a constant K such that*

$$\mathbb{E} \left[\int_{\theta}^T |Z_t^1(\theta) - Z_{\pi(t)}^1(\theta)|^2 dt \right] \leq K \left(1 + \mathbb{E} \left[|X_{\theta}^1(\theta)|^4 \right]^{\frac{1}{2}} \right) |\pi|, \quad (5.1)$$

for all $\theta \in \pi$.

Proof. The proof follows the main lines of the proof of Proposition 4.5 in [4]. The main difference is that the starting time is not $t = 0$ but $t = \theta$. Firstly, we suppose that b, σ, f and g are in C_b^1 . Let us define the processes Λ and M by

$$\Lambda_t := \exp \left(\int_{\theta}^t \partial_y f(\Theta_r^1(\theta)) dr \right),$$

and

$$M_t := 1 + \int_{\theta}^t M_r \partial_z f(\Theta_r^1(\theta)) dW_r,$$

where $\Theta_r^1(\theta) = (r, X_r^1(\theta), Y_r^1(\theta), Z_r^1(\theta), 0)$. We give classically the link between $\nabla^{\theta} X_t^1(\theta) (= \partial X_t^1(\theta) / \partial X_{\theta}^1(\theta))$ and $(D_s X_t^1(\theta))_{\theta \leq s \leq t}$ the Malliavin derivative of $X_t^1(\theta)$. Recall that $X^1(\theta)$ satisfies

$$X_t^1(\theta) = X_{\theta}^1(\theta) + \int_{\theta}^t b(r, X_r^1(\theta)) dr + \int_{\theta}^t \sigma(r, X_r^1(\theta)) dW_r, \quad \theta \leq t \leq T.$$

Therefore, we get

$$\nabla^{\theta} X_t^1(\theta) = 1 + \int_{\theta}^t \nabla b(r, X_r^1(\theta)) \nabla^{\theta} X_r^1(\theta) dr + \int_{\theta}^t \nabla \sigma(r, X_r^1(\theta)) \nabla^{\theta} X_r^1(\theta) dW_r, \quad \theta \leq t \leq T,$$

and for $\theta \leq s \leq t$

$$D_s X_t^1(\theta) = \sigma(s, X_s^1(\theta)) + \int_s^t \nabla b(r, X_r^1(\theta)) D_s X_r^1(\theta) dr + \int_s^t \nabla \sigma(r, X_r^1(\theta)) D_s X_r^1(\theta) dr .$$

Thus, we have

$$D_s X_t^1(\theta) = \nabla^\theta X_t^1(\theta) [\nabla^\theta X_s^1(\theta)]^{-1} \sigma(s, X_s^1(\theta)) . \quad (5.2)$$

Using Malliavin calculus we obtain that a version of $Z^1(\theta)$ is given by $(D_t Y_t^1(\theta))_{t \in [\theta, T]}$. By Itô's formula, we get

$$\Lambda_t M_t Z_t^1(\theta) = \mathbb{E} \left[M_T \left(\Lambda_T \nabla g(X_T^1(\theta)) D_t X_T^1(\theta) + \int_t^T \partial_x f(\Theta_r^1(\theta)) D_t X_r^1(\theta) \Lambda_r dr \right) \middle| \mathcal{F}_t^1(\theta) \right] ,$$

for $t \in [\theta, T]$. Using (5.2), we get

$$\Lambda_t M_t Z_t^1(\theta) = \mathbb{E} \left[M_T \left(\Lambda_T \nabla g(X_T^1(\theta)) \nabla^\theta X_T^1(\theta) + \int_t^T F_r \Lambda_r dr \right) \middle| \mathcal{F}_t^1(\theta) \right] [\nabla^\theta X_t^1(\theta)]^{-1} \sigma(t, X_t^1(\theta)) ,$$

with $F_r = \partial_x f(\Theta_r^1(\theta)) \nabla^\theta X_r^1(\theta)$. Which implies that

$$\Lambda_t M_t Z_t^1(\theta) = \left(\mathbb{E}[G | \mathcal{F}_t^1(\theta)] - M_t \int_\theta^t F_r \Lambda_r dr \right) [\nabla^\theta X_t^1(\theta)]^{-1} \sigma(t, X_t^1(\theta)) ,$$

with $G = M_T \left(\Lambda_T \nabla g(X_T^1(\theta)) \nabla^\theta X_T^1(\theta) + \int_\theta^T F_r \Lambda_r dr \right)$. Since b , σ , f and g have bounded derivatives, we have

$$\mathbb{E}[|G|^p] \leq \infty , \quad p \geq 2 . \quad (5.3)$$

Define $m_r = \mathbb{E}[G | \mathcal{F}_r^1(\theta)]$ for $r \in [\theta, T]$. From (5.3) and Doob's inequality, we have

$$\|m\|_{S^p[\theta, T]} < \infty , \quad p \geq 2 . \quad (5.4)$$

Hence, there exists a process ϕ such that

$$m_r = \mathbb{E}[G | \mathcal{F}_\theta^1(\theta)] + \int_\theta^r \phi_u dW_u , \quad r \in [\theta, T] ,$$

and

$$\|\phi\|_{H^p[\theta, T]} < \infty , \quad p \geq 2 .$$

We define \tilde{Z} by

$$\tilde{Z}_t = (\Lambda_t M_t)^{-1} \left(m_t - M_t \int_\theta^t F_r \Lambda_r dr \right) [\nabla^\theta X_t^1(\theta)]^{-1} .$$

By Itô's formula, we can write

$$\tilde{Z}_t = \tilde{Z}_\theta + \int_\theta^t \alpha_r^1 dr + \int_\theta^t \alpha_r^2 dW_r, \quad \theta \leq r \leq T.$$

Since b , σ , f and g have bounded derivatives, we get from (5.4)

$$\|\tilde{Z}\|_{\mathcal{S}^p[\theta, T]}^p < \infty, \quad p \geq 2, \quad (5.5)$$

and

$$\|\alpha^1\|_{H^p[\theta, T]} + \|\alpha^2\|_{H^p[\theta, T]} < \infty, \quad p \geq 2. \quad (5.6)$$

We now write for $t \in [t_i, t_{i+1})$

$$\mathbb{E}[|Z_t - Z_{t_i}|^2] \leq K(I_{t_i, t}^1 + I_{t_i, t}^2),$$

with

$$\begin{cases} I_{t_i, t}^1 = \mathbb{E}[|\tilde{Z}_t - \tilde{Z}_{t_i}|^2 | \sigma(t_i, X_{t_i}^1(\theta))|^2], \\ I_{t_i, t}^2 = \mathbb{E}[|\tilde{Z}_t|^2 | \sigma(t, X_t^1(\theta)) - \sigma(t_i, X_{t_i}^1(\theta))|^2]. \end{cases}$$

We give an upper bound for each term.

$$\begin{aligned} I_{t_i, t}^1 &= \mathbb{E}\left[\mathbb{E}[|\tilde{Z}_t - \tilde{Z}_{t_i}|^2 | \mathcal{F}_{t_i}^1(\theta)] | \sigma(t_i, X_{t_i}^1(\theta))|^2\right] \\ &\leq K \mathbb{E}\left[\int_{t_i}^{t_{i+1}} (|\alpha_r^1|^2 + |\alpha_r^2|^2) dr | \sigma(t_i, X_{t_i}^1(\theta))|^2\right] \\ &\leq K \mathbb{E}\left[\int_{t_i}^{t_{i+1}} (|\alpha_r^1|^2 + |\alpha_r^2|^2) dr \sup_{t \in [\theta, T]} |\sigma(t, X_t^1(\theta))|^2\right] \end{aligned}$$

which implies

$$\int_{t_i}^{t_{i+1}} I_{t_i, t}^1 dt \leq K|\pi| \mathbb{E}\left[\int_{t_i}^{t_{i+1}} (|\alpha_r^1|^2 + |\alpha_r^2|^2) dr \sup_{t \in [\theta, T]} |\sigma(t, X_t^1(\theta))|^2\right],$$

therefore we have

$$\sum_{i=0, t_i \geq \theta}^{n-1} \int_{t_i}^{t_{i+1}} I_{t_i, t}^1 dt \leq K|\pi| \mathbb{E}\left[\int_\theta^T (|\alpha_r^1|^2 + |\alpha_r^2|^2) dr \sup_{t \in [\theta, T]} |\sigma(t, X_t^1(\theta))|^2\right].$$

From Hölder's inequality and Lipschitz property of σ , we have

$$\sum_{i=0, t_i \geq \theta}^{n-1} \int_{t_i}^{t_{i+1}} I_{t_i, t}^1 dt \leq K|\pi| \mathbb{E}\left[\int_\theta^T (|\alpha_r^1|^4 + |\alpha_r^2|^4) dr\right]^{\frac{1}{2}} \left(1 + \mathbb{E}\left[\sup_{t \in [\theta, T]} |X_t^1(\theta)|^4\right]^{\frac{1}{2}}\right).$$

Using (5.6), we get

$$\begin{aligned} \sum_{i=0, t_i \geq \theta}^{n-1} \int_{t_i}^{t_{i+1}} I_{t_i, t}^1 dt &\leq K|\pi| \left(1 + \mathbb{E}\left[\sup_{t \in [\theta, T]} |X_t^1(\theta)|^4\right]^{\frac{1}{2}}\right) \\ &\leq K|\pi| \left(1 + \mathbb{E}\left[|X_\theta^1(\theta)|^4\right]^{\frac{1}{2}}\right). \end{aligned} \quad (5.7)$$

Since σ is Lipschitz, we get from (5.5)

$$\begin{aligned} I_{t_i, t}^2 &\leq K \left(\mathbb{E} \left[|X_t^1(\theta) - X_{t_i}^1(\theta)|^2 |\tilde{Z}_t|^2 \right] + |\pi|^2 \right) \\ &\leq K \left(\mathbb{E} \left[|\tilde{Z}_t - \tilde{Z}_{t_i}|^2 |X_{t_i}^1(\theta)|^2 \right] + \mathbb{E} \left[|X_t^1(\theta) \tilde{Z}_t - X_{t_i}^1(\theta) \tilde{Z}_{t_i}|^2 \right] + |\pi|^2 \right). \end{aligned}$$

Arguing as above, we obtain

$$\sum_{i=0, t_i \geq \theta}^{n-1} \int_{t_i}^{t_{i+1}} \mathbb{E} \left[|\tilde{Z}_t - \tilde{Z}_{t_i}|^2 |X_{t_i}^1(\theta)|^2 \right] dt \leq K |\pi| \mathbb{E} \left[\sup_{\theta \leq t \leq T} (1 + |X_t^1(\theta)|^4) \right]^{\frac{1}{2}}.$$

Moreover, from Itô's formula, $X^1(\theta) \tilde{Z}$ is a semimartingale of the form

$$X_t^1(\theta) \tilde{Z}_t = X_\theta^1(\theta) \tilde{Z}_\theta + \int_\theta^t \tilde{\alpha}_r^1 dr + \int_\theta^t \tilde{\alpha}_r^2 dW_r,$$

where $\|\tilde{\alpha}^1\|_{H^2[\theta, T]} + \|\tilde{\alpha}^2\|_{H^2[\theta, T]} \leq K(1 + \mathbb{E}[|X_\theta^1(\theta)|^4]^{\frac{1}{4}})$. Therefore, we have

$$\mathbb{E} \left[|X_t^1(\theta) \tilde{Z}_t - X_{t_i}^1(\theta) \tilde{Z}_{t_i}|^2 \right] \leq K \mathbb{E} \left[\int_{t_i}^{t_{i+1}} (|\tilde{\alpha}_r^1|^2 + |\tilde{\alpha}_r^2|^2) dr \right],$$

which implies

$$\sum_{i=0, t_i \geq \theta}^{n-1} \int_{t_i}^{t_{i+1}} \mathbb{E} \left[|X_t^1(\theta) \tilde{Z}_t - X_{t_i}^1(\theta) \tilde{Z}_{t_i}|^2 \right] \leq K |\pi| \mathbb{E} \left[(1 + |X_\theta^1(\theta)|^4) \right]^{\frac{1}{2}}. \quad (5.8)$$

Using (5.7) and (5.8) we get the result.

When b, σ, β, f and g are not in C_b^1 , we can also prove the result by regularization. We first suppose that f and g are in C_b^1 . We consider a density q which is C_b^∞ on \mathbb{R} with a compact support, and we define an approximation $(b^\epsilon, \sigma^\epsilon, \beta^\epsilon)$ of (b, σ, β) in C_b^1 by

$$(b^\epsilon, \sigma^\epsilon, \beta^\epsilon)(t, x) = \frac{1}{\epsilon} \int_{\mathbb{R}} (b, \sigma, \beta)(t, x') q\left(\frac{x - x'}{\epsilon}\right) dx', \quad (t, x) \in [0, T] \times \mathbb{R}.$$

We then use the convergence of $(X^{1, \epsilon}(\theta), Y^{1, \epsilon}(\theta), Z^{1, \epsilon}(\theta))$ to $(X^1(\theta), Y^1(\theta), Z^1(\theta))$ and we get the result. Next we assume that f and g are not C_b^1 and we consider for that f^ϵ and g^ϵ which are defined as previously and we get the result. \square

Using the link between X^0 and $X_\theta^1(\theta)$, we obtain that the bound (5.1) is actually uniform in θ .

Corollary 5.1. *Under **(HF)**, **(HFD)**, **(HBL)** and **(HBLD)**, there exists a constant K such that*

$$\mathbb{E} \left[\int_\theta^T |Z_t^1(\theta) - Z_{\pi(t)}^1(\theta)|^2 dt \right] \leq K |\pi|, \quad (5.9)$$

for all $\theta \in \pi$.

Proof. We notice that from the Lipschitz property of β we have

$$\begin{aligned}\mathbb{E}\left[|X_\theta^1(\theta)|^4\right] &= \mathbb{E}\left[|X_\theta^0 + \beta(\theta, X_\theta^0)|^4\right] \\ &\leq K\left(1 + \mathbb{E}\left[\sup_{t \in [0, T]} |X_t^0|^4\right]\right) < \infty.\end{aligned}$$

Combining this result with (5.1), we get (5.9) \square

We now study the regularity of Z^0 .

Proposition 5.2. *Under (HF), (HFD), (HBL) and (HBLD), there exists a constant K such that we have*

$$\mathbb{E}\left[\int_0^T |Z_t^0 - Z_{\pi(t)}^0|^2 dt\right] \leq K|\pi|.$$

Proof. The proof is similar to the previous one. The only difference is that BSDE (2.6) involves Y^1 . We denote $\Theta_r^0 = (r, X_r^0, Y_r^0, Z_r^0, Y_r^1(r) - Y_r^0)$. Firstly, we suppose that b, σ, β, f and g are in C_b^1 . We recall that

$$Y_t^0 = g(X_T^0) + \int_t^T f(\Theta_s^0) ds - \int_t^T Z_s^0 dW_s.$$

Therefore, for $0 \leq r \leq t \leq T$, we have

$$\begin{aligned}D_r Y_t^0 &= \nabla g(X_T^0) D_r X_T^0 + \int_t^T \left(\nabla_x f(\Theta_s^0) D_r X_s^0 + (\nabla_y - \nabla_u) f(\Theta_s^0) D_r Y_s^0 \right. \\ &\quad \left. + \nabla_z f(\Theta_s^0) D_r Z_s^0 + \nabla_u f(\Theta_s^0) D_r Y_s^1(s) \right) dr - \int_t^T D_r Z_s^0 dW_s.\end{aligned}$$

where DX_r^0, DY_r^0, DZ_r^0 and $DY_r^1(r)$ denote the Malliavin derivatives of X_r^0, Y_r^0, Z_r^0 and $Y_r^1(r)$ for $r \in [0, T]$. Using Malliavin calculus, we obtain that a version of Z^0 is given by $(D_t Y_t^0)_{t \in [0, T]}$. By Itô's formula, we get

$$\Lambda_t M_t Z_t = \mathbb{E}\left[M_T \left(\Lambda_T \nabla g(X_T^0) D_t X_T^0 + \int_t^T (\nabla_x f(\Theta_r^0) D_t X_r^0 + \nabla_u f(\Theta_r^0) D_t Y_r^1(r)) \Lambda_r dr \right) \middle| \mathcal{F}_t^0 \right],$$

where $\Lambda_t = \exp(\int_0^t (\nabla_y - \nabla_u) f(\Theta_r^0) dr)$ and $M_t = 1 + \int_0^t M_r \nabla_z f(\Theta_r^0) dW_r$. Denote by $\nabla X_t^0 := \frac{\partial X_t^0}{\partial X_0^0}$ and $\nabla X_t^1(\theta) := \frac{\partial X_t^1(\theta)}{\partial X_0^1(\theta)}$ for $0 \leq t \leq \theta \leq T$. We then have for $r \leq s \leq T$

$$D_r X_s^1(s) = (1 + \nabla_x \beta(s, X_s^0)) D_r X_s^0 = (1 + \nabla_x \beta(s, X_s^0)) \nabla X_s^0 \sigma(r, X_r^0) [\nabla X_r^0]^{-1},$$

thus we can see that $D_r X_s^1(s) = \nabla X_s^1(s) \sigma(r, X_r^0) [\nabla X_r^0]^{-1}$. Therefore, we get by writing the SDEs satisfied by $(D_r X_s^1(\theta))_{s \in [\theta, T]}$ for $r \leq \theta$, and $(\nabla X_s^1(\theta))_{s \in [\theta, T]}$

$$D_r X_s^1(\theta) = \nabla X_s^1(\theta) [\nabla X_r^0]^{-1} \sigma(r, X_r^0), \quad r \leq \theta \leq s.$$

Writing the BSDEs satisfied by $(D_r Y_s^1(\theta))_{s \in [\theta, T]}$ for $r \leq \theta$ and $(\nabla Y_s^1(\theta))_{s \in [\theta, T]}$, and using the previous equality, we get

$$D_r Y_s^1(s) = \nabla Y_s^1(s) [\nabla X_r^0]^{-1} \sigma(r, X_r^0), \quad s \leq \theta.$$

Which implies

$$\Lambda_t M_t Z_t = \mathbb{E} \left[M_T \left(\Lambda_T \nabla g(X_T^0) \nabla X_T^0 + \int_t^T F_r \Lambda_r dr \right) \right] [\nabla X_t^0]^{-1} \sigma(t, X_t^0),$$

with $F_r = \nabla_x f(\Theta_r^0) \nabla X_r^0 + \nabla_u f(\Theta_r^0) \nabla Y_r^1(r)$. We can write

$$\Lambda_t M_t Z_t = \left(\mathbb{E}[G | \mathcal{F}_t^0] - \int_0^t M_r F_r \Lambda_r dr \right) [\nabla X_t^0]^{-1} \sigma(t, X_t^0),$$

with $G = M_T (\Lambda_T \nabla g(X_T^0) \nabla X_T^0 + \int_0^T F_r \Lambda_r dr)$. Since b , σ , f and g have bounded derivatives, we have

$$\mathbb{E}[|G|^p] \leq \infty, \quad p \geq 2. \quad (5.10)$$

Define $m_r = \mathbb{E}[G | \mathcal{F}_r^0]$ for $r \in [0, T]$. From (5.10) and Doob's inequality, we have

$$\|m\|_{S^p[0, T]} \leq \infty, \quad p \geq 2. \quad (5.11)$$

Hence, there exists a process ϕ such that

$$m_r = \mathbb{E}[G] + \int_0^r \phi_u dW_u, \quad r \in [0, T],$$

and

$$\|\phi\|_{H^p[0, T]} \leq \infty, \quad p \geq 2.$$

We define \tilde{Z} by

$$\tilde{Z}_t = (\Lambda_t M_t)^{-1} \left(m_t - M_t \int_0^t F_r \Lambda_r dr \right) [\nabla X_t^0]^{-1}, \quad t \in [0, T].$$

By Itô's formula, we can write

$$\tilde{Z}_t = \tilde{Z}_0 + \int_0^t \alpha_r^1 ds + \int_0^t \alpha_r^2 dW_r, \quad t \in [0, T].$$

Using the fact that b , σ , f and g have bounded derivatives and (5.11), we get

$$\|\tilde{Z}\|_{S^p[0, T]}^p < \infty, \quad p \geq 2,$$

and

$$\|\alpha^1\|_{H^p[0, T]} + \|\alpha^2\|_{H^p[0, T]} < \infty, \quad p \geq 2. \quad (5.12)$$

We now write for $t \in [t_i, t_{i+1})$

$$\mathbb{E}[|Z_t^0 - Z_{t_i}^0|^2] \leq K(I_{t_i,t}^1 + I_{t_i,t}^2),$$

with

$$\begin{cases} I_{t_i,t}^1 = \mathbb{E}[|\tilde{Z}_t - \tilde{Z}_{t_i}|^2 |\sigma(t_i, X_{t_i}^0)|^2], \\ I_{t_i,t}^2 = \mathbb{E}[|\tilde{Z}_t|^2 |\sigma(t, X_t^0) - \sigma(t_i, X_{t_i}^0)|^2]. \end{cases}$$

As previously we give an upper bound for each term.

$$I_{t_i,t}^1 \leq K \mathbb{E} \left[\int_{t_i}^{t_{i+1}} (|\alpha_r^1|^2 + |\alpha_r^2|^2) dr \sup_{t \in [0, T]} |\sigma(t, X_t^0)|^2 \right].$$

From Hölder's inequality and Lipschitz property of σ , we have

$$\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} I_{t_i,t}^1 dt \leq K |\pi| \mathbb{E} \left[\int_0^T (|\alpha_r^1|^4 + |\alpha_r^2|^4) dr \right]^{\frac{1}{2}} \left(1 + \mathbb{E} \left[\sup_{t \in [0, T]} |X_t^0|^4 \right]^{\frac{1}{2}} \right).$$

Using (5.12), we get

$$\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} I_{t_i,t}^1 dt \leq K |\pi|.$$

Since σ is Lipschitz, we get

$$I_{t_i,t}^2 \leq K \left(\mathbb{E}[|\tilde{Z}_t - \tilde{Z}_{t_i}|^2 |X_{t_i}^0|^2] + \mathbb{E}[|X_t^0 \tilde{Z}_t - X_{t_i}^0 \tilde{Z}_{t_i}|^2] + |\pi|^2 \right).$$

Arguing as above, we obtain

$$\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \mathbb{E}[|\tilde{Z}_t - \tilde{Z}_{t_i}|^2 |X_{t_i}^0|^2] dt \leq K |\pi| \left(1 + \mathbb{E} \left[\sup_{t \in [0, T]} |X_t^0|^4 \right]^{\frac{1}{2}} \right).$$

Moreover, $X^0 \tilde{Z}$ is a semimartingale of the form

$$X_t^0 \tilde{Z}_t = X_0^0 \tilde{Z}_0 + \int_0^t \tilde{\alpha}_r^1 dr + \int_0^t \tilde{\alpha}_r^2 dW_r$$

where $\|\tilde{\alpha}^1\|_{H^2[0, T]} + \|\tilde{\alpha}^2\|_{H^2[0, T]} \leq K$ and we have

$$\mathbb{E}[|X_t^0 \tilde{Z}_t - X_{t_i}^0 \tilde{Z}_{t_i}|^2] \leq K \mathbb{E} \int_{t_i}^{t_{i+1}} (|\tilde{\alpha}_r^1|^2 + |\tilde{\alpha}_r^2|^2) dr,$$

which implies

$$\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \mathbb{E}[|X_t^0 \tilde{Z}_t - X_{t_i}^0 \tilde{Z}_{t_i}|^2] \leq K |\pi|.$$

When b , σ , f and g are not C_b^1 , we can also prove the result by regularization as for Proposition 5.1. \square

5.2 Error estimates for the recursive system of BSDEs

We first state an estimate of the approximation error for (Y^1, Z^1) .

Proposition 5.3. *Under **(HF)**, **(HFD)**, **(HBL)** and **(HBLD)**, we have the following estimate*

$$\sup_{\theta \in [0, T]} \left\{ \sup_{t \in [\theta, T]} \mathbb{E} \left[|Y_t^1(\theta) - Y_{\pi(t)}^{1, \pi}(\pi(\theta))|^2 \right] + \mathbb{E} \left[\int_{\theta}^T |Z_s^1(\theta) - Z_{\pi(s)}^{1, \pi}(\pi(\theta))|^2 ds \right] \right\} \leq K|\pi|,$$

for some constant K which does not depend on π .

Proof. Fix $\theta \in [0, T]$ and $t \in [\theta, T]$. We then have

$$\begin{aligned} \mathbb{E} \left[|Y_t^1(\theta) - Y_{\pi(t)}^{1, \pi}(\pi(\theta))|^2 \right] &\leq 2 \mathbb{E} \left[|Y_t^1(\theta) - Y_t^1(\pi(\theta))|^2 \right] \\ &\quad + 2 \mathbb{E} \left[|Y_t^1(\pi(\theta)) - Y_{\pi(t)}^{1, \pi}(\pi(\theta))|^2 \right]. \end{aligned} \quad (5.13)$$

We study separately the two terms of right hand side.

Define $\delta X_t^1(\theta) := X_t^1(\theta) - X_t^1(\pi(\theta))$, $\delta Y_t^1(\theta) := Y_t^1(\theta) - Y_t^1(\pi(\theta))$ and $\delta Z_t^1(\theta) := Z_t^1(\theta) - Z_t^1(\pi(\theta))$. Applying Itô's formula, we get

$$\begin{aligned} |\delta Y_T^1(\theta)|^2 - |\delta Y_t^1(\theta)|^2 &= 2 \int_t^T \delta Y_s^1(\theta) \left[f(\Theta_s^1(\pi(\theta))) - f(\Theta_s^1(\theta)) \right] ds \\ &\quad + 2 \int_t^T \delta Y_s^1(\theta) \delta Z_s^1(\theta) dW_s + \int_t^T |\delta Z_s^1(\theta)|^2 ds, \end{aligned}$$

where $\Theta_r^1(\theta) = (s, X_s^1(\theta), Y_s^1(\theta), Z_s^1(\theta), 0)$. From **(HBL)** and **(HBLD)**, there exists a constant K such that

$$\begin{aligned} \mathbb{E} [|\delta Y_t^1(\theta)|^2] &\leq K \left(\mathbb{E} [|\delta X_T^1(\theta)|^2] + \mathbb{E} \left[\int_t^T |\delta Y_s^1(\theta)| |\delta X_s^1(\theta)| ds \right] + \mathbb{E} \left[\int_t^T |\delta Y_s^1(\theta)|^2 ds \right] \right. \\ &\quad \left. + \mathbb{E} \left[\int_t^T |\delta Y_s^1(\theta)| |\delta Z_s^1(\theta)| ds \right] \right) - \mathbb{E} \left[\int_t^T |\delta Z_s^1(\theta)|^2 ds \right]. \end{aligned}$$

Using the inequality $2ab \leq a^2/\eta + \eta b^2$ for $a, b \in \mathbb{R}$ and $\eta > 0$, we can see that there exists a constant K such that

$$\begin{aligned} \mathbb{E} [|\delta Y_t^1(\theta)|^2] + \mathbb{E} \left[\int_t^T |\delta Z_s^1(\theta)|^2 ds \right] &\leq K \left(\mathbb{E} [|\delta X_T^1(\theta)|^2] + \int_t^T \mathbb{E} [|\delta Y_s^1(\theta)|^2] ds \right. \\ &\quad \left. + \mathbb{E} \left[\int_t^T |\delta X_s^1(\theta)|^2 ds \right] \right). \end{aligned} \quad (5.14)$$

From (4.4) and Gronwall's lemma, we get

$$\mathbb{E} [|Y_t^1(\theta) - Y_t^1(\pi(\theta))|^2] \leq K|\pi|, \quad (5.15)$$

for some constant K which neither depends on π nor on θ .

We now study the second term of the right hand side of (5.13). Using the same argument as in the proof of Theorem 3.1 in [3], we get from the regularity of Z^1 given by Corollary 5.1

$$\mathbb{E}\left[|Y_t^1(\pi(\theta)) - Y_{\pi(t)}^{1,\pi}(\pi(\theta))|^2\right] \leq K|\pi|, \quad (5.16)$$

for some constant K which neither depends on π nor on θ . This last inequality with (5.13) and (5.15) gives

$$\sup_{\theta \in [0, T]} \left\{ \sup_{t \in [\theta, T]} \mathbb{E}\left[|Y_t^1(\theta) - Y_{\pi(t)}^{1,\pi}(\pi(\theta))|^2\right] \right\} \leq K|\pi|.$$

We now turn to the error on the term $Z^1(\theta)$. We first use the inequality

$$\begin{aligned} \mathbb{E}\left[\int_{\theta}^T |Z_t^1(\theta) - Z_{\pi(t)}^{1,\pi}(\pi(\theta))|^2 dt\right] &\leq 2 \mathbb{E}\left[\int_{\theta}^T |Z_t^1(\pi(\theta)) - Z_{\pi(t)}^{1,\pi}(\pi(\theta))|^2 dt\right] \\ &\quad + 2 \mathbb{E}\left[\int_{\theta}^T |\delta Z_t^1(\theta)|^2 dt\right]. \end{aligned} \quad (5.17)$$

Using (5.14) and (5.15) with $t = \theta$, we get

$$\mathbb{E}\left[\int_{\theta}^T |\delta Z_s^1(\theta)|^2 ds\right] \leq K|\pi|, \quad (5.18)$$

for some constant K which neither depends on π nor on θ . The other term in the right hand side of (5.17) is the classical error in an approximation of BSDE. Therefore, using Corollary 5.1 and (5.16), we have

$$\mathbb{E}\left[\int_{\theta}^T |Z_t^1(\pi(\theta)) - Z_{\pi(t)}^{1,\pi}(\pi(\theta))|^2 dt\right] \leq K|\pi|, \quad (5.19)$$

for some constant K which neither depends on π nor on θ . Combining (5.17), (5.18) and (5.19), we get

$$\mathbb{E}\left[\int_{\theta}^T |Z_t^1(\theta) - Z_{\pi(t)}^{1,\pi}(\pi(\theta))|^2 dt\right] \leq K|\pi|.$$

□

We now turn to the estimation of the error between (Y^0, Z^0) and its Euler scheme (3.5). Since this scheme involves the approximation $Y^{1,\pi}$ of Y^1 , we first need to introduce an intermediary scheme involving the "true" value of the process Y^1 . We therefore consider the scheme $(\tilde{Y}^{0,\pi}, \tilde{Z}^{0,\pi})$ defined by

$$\begin{cases} \tilde{Y}_T^{0,\pi} &= g(X_T^{0,\pi}), \\ \tilde{Y}_{t_{i-1}}^{0,\pi} &= \mathbb{E}_{i-1}^0[\tilde{Y}_{t_i}^{0,\pi}] + f(t_{i-1}, X_{t_{i-1}}^{0,\pi}, \tilde{Y}_{t_{i-1}}^{0,\pi}, \tilde{Z}_{t_{i-1}}^{0,\pi}, Y_{t_{i-1}}^1(t_{i-1}) - \tilde{Y}_{t_{i-1}}^{0,\pi}) \Delta t_i^\pi, \\ \tilde{Z}_{t_{i-1}}^{0,\pi} &= \frac{1}{\Delta t_i^\pi} \mathbb{E}_{i-1}^0[\tilde{Y}_{t_i}^{0,\pi} \Delta W_i^\pi], \quad 1 \leq i \leq n. \end{cases} \quad (5.20)$$

Using the regularity result of Proposition 5.2 and the same arguments as in the proof of Theorem 3.1 in [3], we get under **(HF)**, **(HFD)**, **(HBL)** and **(HBLD)**

$$\sup_{t \in [0, T]} \mathbb{E} \left[|Y_t^0 - \tilde{Y}_{\pi(t)}^{0, \pi}|^2 \right] + \mathbb{E} \left[\int_0^T |Z_t^0 - \tilde{Z}_{\pi(t)}^{0, \pi}|^2 dt \right] \leq K|\pi|. \quad (5.21)$$

With this inequality, we get the following estimate for the error between (Y^0, Z^0) and the Euler scheme (3.5).

Proposition 5.4. *Under **(HF)**, **(HFD)**, **(HBL)** and **(HBLD)**, we have the following estimate*

$$\sup_{t \in [0, T]} \mathbb{E} \left[|Y_t^0 - Y_{\pi(t)}^{0, \pi}|^2 \right] + \mathbb{E} \left[\int_0^T |Z_t^0 - Z_{\pi(t)}^{0, \pi}|^2 dt \right] \leq K|\pi|,$$

for some constant K which does not depend on π .

Proof. We first remark that

$$\begin{cases} \sup_{t \in [0, T]} \mathbb{E} \left[|Y_t^0 - Y_{\pi(t)}^{0, \pi}|^2 \right] \leq 2 \sup_{t \in [0, T]} \mathbb{E} \left[|Y_t^0 - \tilde{Y}_{\pi(t)}^{0, \pi}|^2 \right] + 2 \sup_{t \in [0, T]} \mathbb{E} \left[|Y_{\pi(t)}^{0, \pi} - \tilde{Y}_{\pi(t)}^{0, \pi}|^2 \right], \\ \mathbb{E} \left[\int_0^T |Z_t^0 - Z_{\pi(t)}^{0, \pi}|^2 dt \right] \leq 2 \mathbb{E} \left[\int_0^T |Z_t^0 - \tilde{Z}_{\pi(t)}^{0, \pi}|^2 dt \right] + 2 \mathbb{E} \left[\int_0^T |Z_{\pi(t)}^{0, \pi} - \tilde{Z}_{\pi(t)}^{0, \pi}|^2 dt \right]. \end{cases}$$

Using (5.21), we only need to study $\sup_{t \in [0, T]} \mathbb{E} [|Y_{\pi(t)}^{0, \pi} - \tilde{Y}_{\pi(t)}^{0, \pi}|^2]$ and $\mathbb{E} [\int_0^T |Z_{\pi(t)}^{0, \pi} - \tilde{Z}_{\pi(t)}^{0, \pi}|^2 dt]$. To this end, we introduce for all $0 \leq i \leq n-1$ the continuous schemes

$$\begin{cases} Y_t^{0, \pi} = Y_{t_i}^{0, \pi} - (t - t_i) f(t_i, X_{t_i}^{0, \pi}, Y_{t_i}^{0, \pi}, Z_{t_i}^{0, \pi}, Y_{t_i}^{1, \pi}(t_i) - Y_{t_i}^{0, \pi}) + \int_{t_i}^t \underline{Z}_s^{0, \pi} dW_s, \\ \tilde{Y}_t^{0, \pi} = \tilde{Y}_{t_i}^{0, \pi} - (t - t_i) f(t_i, X_{t_i}^{0, \pi}, \tilde{Y}_{t_i}^{0, \pi}, \tilde{Z}_{t_i}^{0, \pi}, Y_{t_i}^1(t_i) - \tilde{Y}_{t_i}^{0, \pi}) + \int_{t_i}^t \tilde{\underline{Z}}_s^{0, \pi} dW_s \end{cases}$$

for $t \in [t_i, t_{i+1})$. Let $i \in \{0, \dots, n-1\}$ be fixed, and set $\delta Y_t := Y_t^{0, \pi} - \tilde{Y}_t^{0, \pi}$, $\delta Z_t := Z_t^{0, \pi} - \tilde{Z}_t^{0, \pi}$, $\delta \underline{Z}_t := \underline{Z}_t^{0, \pi} - \tilde{\underline{Z}}_t^{0, \pi}$ and $\delta f_t := f(t_i, X_{t_i}^{0, \pi}, Y_{t_i}^{0, \pi}, Z_{t_i}^{0, \pi}, Y_{t_i}^{1, \pi}(t_i) - Y_{t_i}^{0, \pi}) - f(t_i, X_{t_i}^{0, \pi}, \tilde{Y}_{t_i}^{0, \pi}, \tilde{Z}_{t_i}^{0, \pi}, Y_{t_i}^1(t_i) - \tilde{Y}_{t_i}^{0, \pi})$ for $t \in [t_i, t_{i+1})$. By Itô's formula, we compute that

$$A_t := \mathbb{E} |\delta Y_t|^2 + \int_t^{t_{i+1}} \mathbb{E} |\delta \underline{Z}_s|^2 ds - \mathbb{E} |\delta Y_{t_{i+1}}|^2 = 2 \int_t^{t_{i+1}} \mathbb{E} [\delta Y_s \delta f_s] ds, \quad t_i \leq t \leq t_{i+1}.$$

Let $\alpha > 0$ be a constant to be chosen later on. From the Lipschitz property of f , together with the inequality $2ab \leq \alpha a^2 + b^2/\alpha$, this provides

$$A_t \leq \alpha \int_t^{t_{i+1}} \mathbb{E} |\delta Y_s|^2 ds + \frac{K}{\alpha} \int_t^{t_{i+1}} \mathbb{E} \left[|\delta Y_{t_i}|^2 + |\delta Z_i|^2 + |Y_{t_i}^1(t_i) - Y_{t_i}^{1, \pi}(t_i)|^2 \right] ds.$$

Using Proposition 5.3, we get

$$A_t \leq \alpha \int_t^{t_{i+1}} \mathbb{E} |\delta Y_s|^2 ds + \frac{K}{\alpha} |\pi| \mathbb{E} |\delta Y_{t_i}|^2 + \frac{K}{\alpha} \int_t^{t_{i+1}} \mathbb{E} |\delta Z_i|^2 ds + \frac{K}{\alpha} |\pi|^2.$$

We can write

$$\mathbb{E}|\delta Y_t|^2 \leq \mathbb{E}|\delta Y_t|^2 + \int_t^{t_{i+1}} \mathbb{E}|\delta \underline{Z}_s|^2 ds \leq \alpha \int_t^{t_{i+1}} \mathbb{E}|\delta Y_s|^2 ds + B_i, \quad (5.22)$$

where

$$B_i := \mathbb{E}|\delta Y_{t_{i+1}}|^2 + \frac{K}{\alpha} |\pi| \mathbb{E}|\delta Z_i|^2 + \frac{K}{\alpha} |\pi| \mathbb{E}|\delta Y_{t_i}|^2 + \frac{K}{\alpha} |\pi|^2.$$

By Gronwall's lemma, this shows that $\mathbb{E}|\delta Y_t|^2 \leq B_i e^{\alpha|\pi|}$ for $t_i \leq t < t_{i+1}$, which plugged in the second inequality of (5.22) provides

$$\mathbb{E}|\delta Y_t|^2 + \int_t^{t_{i+1}} \mathbb{E}|\delta \underline{Z}_s|^2 ds \leq B_i \left(1 + \alpha|\pi| e^{\alpha|\pi|}\right). \quad (5.23)$$

Interpreting $Z_{t_i}^{0,\pi}$ (resp. $\tilde{Z}_{t_i}^{0,\pi}$) as the projection of $\underline{Z}^{0,\pi}$ (resp. $\tilde{\underline{Z}}^{0,\pi}$) in $H_{\mathbb{F}}^2[t_i, t_{i+1}]$ on the set of constant processes, we have

$$\int_{t_i}^{t_{i+1}} \mathbb{E}|\delta Z_i|^2 ds \leq \int_{t_i}^{t_{i+1}} \mathbb{E}|\delta \underline{Z}_s|^2 ds. \quad (5.24)$$

Applying (5.23) for $t = t_i$ and $\alpha = 2K$, and using the previous inequality, we get

$$\mathbb{E}|\delta Y_{t_i}|^2 + k_1(\pi) \int_{t_i}^{t_{i+1}} \mathbb{E}|\delta \underline{Z}_s|^2 ds \leq k_2(\pi) \mathbb{E}|\delta Y_{t_{i+1}}|^2 + k_3(\pi) |\pi|^2, \quad 0 \leq i \leq n-1,$$

where $k_1(\pi) = \frac{\frac{1}{2} - K|\pi|e^{2K|\pi|}}{1 - \frac{|\pi|}{2} - K|\pi|^2 e^{2K|\pi|}}$, $k_2(\pi) = \frac{1 + 2K|\pi|e^{2K|\pi|}}{1 - \frac{|\pi|}{2} - K|\pi|^2 e^{2K|\pi|}}$ and $k_3(\pi) = \frac{\frac{1}{2} + K|\pi|e^{2K|\pi|}}{1 - \frac{|\pi|}{2} - K|\pi|^2 e^{2K|\pi|}}$. Since for small $|\pi|$ we have $k_1(\pi) \geq 0$, we get

$$\mathbb{E}|\delta Y_{t_i}|^2 \leq k_2(\pi) \mathbb{E}|\delta Y_{t_{i+1}}|^2 + k_3(\pi) |\pi|^2, \quad 0 \leq i \leq n-1,$$

for $|\pi|$ small enough.

Iterating this inequality, we get

$$\mathbb{E}|\delta Y_{t_i}|^2 \leq k_2(\pi)^{\frac{1}{|\pi|}} \mathbb{E}|\delta Y_{t_n}|^2 + |\pi|^2 k_3(\pi) \sum_{j=i}^n k_2(\pi)^{j-i}.$$

Since $k_2(\pi) \geq 1$ and $\delta Y_{t_n} = 0$, we get for small $|\pi|$

$$\mathbb{E}|\delta Y_{t_i}|^2 \leq |\pi| k_3(\pi) k_2(\pi)^{\frac{1}{|\pi|}} \leq K|\pi|, \quad 0 \leq i \leq n, \quad (5.25)$$

which gives

$$\sup_{t \in [0, T]} \mathbb{E} \left[\left| Y_{\pi(t)}^{0,\pi} - \tilde{Y}_{\pi(t)}^{0,\pi} \right|^2 \right] \leq K|\pi|.$$

Summing up the inequality (5.23) with $t = t_i$ and $\alpha = 2K$ and using (5.24), we get

$$\begin{aligned} \left(\frac{1}{2} - K|\pi|e^{2K|\pi|}\right) \int_0^T \mathbb{E}|Z_{\pi(s)}^{0,\pi} - \tilde{Z}_{\pi(s)}^{0,\pi}|^2 ds &\leq 2K|\pi|e^{2K|\pi|} \sum_{i=1}^{n-1} \mathbb{E}|\delta Y_{t_i}|^2 + (1 + 2K|\pi|)\mathbb{E}|\delta Y_{t_n}|^2 \\ &+ \left(\frac{1}{2} + K|\pi|e^{2K|\pi|}\right) \left(|\pi| + |\pi| \sum_{i=0}^{n-1} \mathbb{E}|\delta Y_{t_i}|^2\right). \end{aligned}$$

Using (5.25), we get for $|\pi|$ small enough

$$\int_0^T \mathbb{E}|Z_{\pi(s)}^{0,\pi} - \tilde{Z}_{\pi(s)}^{0,\pi}|^2 ds \leq K|\pi|.$$

□

5.3 Error estimate for the BSDE with a jump

We now give an error estimate of the approximation scheme for the BSDE with a jump.

Theorem 5.1. *Under (HF), (HFD), (HBL) and (HBLD), we have the following error estimate for the approximation scheme*

$$\sup_{t \in [0, T]} \mathbb{E} \left[|Y_t - Y_t^\pi|^2 \right] + \mathbb{E} \left[\int_0^T |Z_t - Z_t^\pi|^2 dt \right] + \mathbb{E} \left[\int_0^T |U_t - U_t^\pi|^2 \lambda_t dt \right] \leq K|\pi|,$$

for some constant K which does not depend on π .

Proof.

Step 1. *Error for the variable Y .* Fix $t \in [0, T]$. From Theorem 2.1 and (3.6), we have

$$\mathbb{E} \left[|Y_t - Y_t^\pi|^2 \right] = \mathbb{E} \left[|Y_t^0 - Y_{\pi(t)}^{0,\pi}|^2 \mathbf{1}_{t < \tau} \right] + \mathbb{E} \left[|Y_t^1(\tau) - Y_{\pi(t)}^{1,\pi}(\pi(\tau))|^2 \mathbf{1}_{t \geq \tau} \right].$$

Using (DH), we get

$$\begin{aligned} \mathbb{E} \left[|Y_t - Y_t^\pi|^2 \right] &\leq \mathbb{E} \left[|Y_t^0 - Y_{\pi(t)}^{0,\pi}|^2 \right] + \int_0^T \mathbb{E} \left[|Y_t^1(\theta) - Y_{\pi(t)}^{1,\pi}(\pi(\theta))|^2 \mathbf{1}_{t \geq \theta} \gamma_T(\theta) \right] d\theta \\ &\leq K \left(\mathbb{E} \left[|Y_t^0 - Y_{\pi(t)}^{0,\pi}|^2 \right] + \sup_{\theta \in [0, T]} \sup_{s \in [\theta, T]} \mathbb{E} \left[|Y_s^1(\theta) - Y_{\pi(s)}^{1,\pi}(\pi(\theta))|^2 \right] \right). \end{aligned}$$

Using Propositions 5.3 and 5.4, and since t is arbitrary chosen in $[0, T]$, we get

$$\sup_{t \in [0, T]} \mathbb{E} \left[|Y_t - Y_t^\pi|^2 \right] \leq K|\pi|,$$

for some constant K which does not depend on π .

Step 2. *Error estimate for the variable Z .* From Theorem 2.1 and (3.6), we have

$$\mathbb{E} \left[\int_0^T |Z_t - Z_t^\pi|^2 dt \right] = \mathbb{E} \left[\int_0^{T \wedge \tau} |Z_t^0 - Z_{\pi(t)}^{0,\pi}|^2 dt \right] + \mathbb{E} \left[\int_{T \wedge \tau}^T |Z_t^1(\tau) - Z_{\pi(t)}^{1,\pi}(\pi(\tau))|^2 dt \right].$$

Using **(DH)**, we get

$$\begin{aligned} \mathbb{E} \left[\int_0^T |Z_t - Z_t^\pi|^2 dt \right] &= \int_0^T \int_0^\theta \mathbb{E} \left[|Z_t^0 - Z_{\pi(t)}^{0,\pi}|^2 \gamma_T(\theta) \right] dt d\theta \\ &\quad + \int_0^T \int_\theta^T \mathbb{E} \left[|Z_t^1(\theta) - Z_{\pi(t)}^{1,\pi}(\pi(\theta))|^2 \gamma_T(\theta) \right] dt d\theta . \\ &\leq K \left(\mathbb{E} \left[\int_0^T |Z_t^0 - Z_{\pi(t)}^{0,\pi}|^2 dt \right] + \sup_{\theta \in [0,T]} \mathbb{E} \left[\int_\theta^T |Z_t^1(\theta) - Z_{\pi(t)}^{1,\pi}(\pi(\theta))|^2 dt \right] \right) . \end{aligned}$$

From Propositions 5.3 and 5.4, we get

$$\mathbb{E} \left[\int_0^T |Z_t - Z_t^\pi|^2 dt \right] \leq K |\pi| ,$$

for some constant K which does not depend on π .

Step 3. *Error estimate for the variable U .* From Theorem 2.1 and (3.6), we have

$$\mathbb{E} \left[\int_0^T |U_t - U_t^\pi|^2 \lambda_t dt \right] \leq K \mathbb{E} \left[\int_0^T \left(|Y_t^1(t) - Y_{\pi(t)}^{1,\pi}(\pi(t))|^2 + |Y_t^0 - Y_{\pi(t)}^{0,\pi}|^2 \right) \lambda_t dt \right] .$$

Using **(HBI)**, we get

$$\mathbb{E} \left[\int_0^T |U_t - U_t^\pi|^2 \lambda_t dt \right] \leq K \left(\sup_{\theta \in [0,T]} \sup_{t \in [\theta,T]} \mathbb{E} \left[|Y_t^1(\theta) - Y_{\pi(t)}^{1,\pi}(\pi(\theta))|^2 \right] + \sup_{t \in [0,T]} \mathbb{E} \left[|Y_t^0 - Y_{\pi(t)}^{0,\pi}|^2 \right] \right) .$$

Combining this last inequality with Propositions 5.3 and 5.4, we get the result. \square

Remark 5.1. Our decomposition approach allows us to suppose that the jump coefficient β is only Lipschitz continuous. We do not need to impose any regularity condition on β and any ellipticity assumption on $I_d + \nabla \beta$ as done in [4] in the case where μ is a Poisson random measure independent of W .

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